AN INTRODUCTION TO SYSTEM-THEORETIC METHODS FOR MODEL REDUCTION
Part I: Balancing-based Methods

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Special Semester on
“Model and dimension reduction in uncertain and dynamic systems”
ICERM at Brown University
1. Introduction

2. Model Reduction by Projection

3. Balanced Truncation

4. Final Remarks
1. Introduction
   Application Areas
   Motivation
   Model Reduction for Dynamical Systems
   Basics of Systems and Control Theory
   Realization Theory for Linear Systems
   Qualitative and Quantitative Study of the Approximation Error

2. Model Reduction by Projection

3. Balanced Truncation

4. Final Remarks
**Problem**

*Given a physical problem with dynamics described by the states $x \in \mathbb{R}^n$, where $n$ is the dimension of the state space.*
Introduction
Model Reduction — Abstract Definition

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Because of redundancies, complexity, etc., we want to describe the dynamics of the system using a reduced number of states.

This is the task of model reduction (also: dimension reduction, order reduction).
Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order $N$, where

- input = output of plant,
- output = input of plant.

Modern (LQG-/H$_2$-/H$_\infty$-) control design: $N \geq n$. 

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\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
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\begin{align*}
\dot{v} &= Ev + Fy \\
u &= Hv + Ky
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- real-time constraints,
- increasing fragility for larger $N$. 

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$\implies$ reduce order of plant ($n$) and/or controller ($N$).

Standard MOR techniques in systems and control: balanced truncation and related methods.
Progressive miniaturization: Moore’s Law states that the number of on-chip transistors doubles each 12 (now: 18) months.
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Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when

- decoupling large linear subcircuits,
- modeling transmission lines (interconnect, powergrid), parasitic effects,
- modeling pin packages in VLSI chips,
- modeling circuit elements described by Maxwell’s equation using partial element equivalent circuits (PEEC).
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Standard MOR techniques in circuit simulation: **Krylov subspace** / **Padé approximation** / **rational interpolation methods**.
Resolving complex 3D geometries ⇒ millions of degrees of freedom.

Analysis of elastic deformations requires many simulation runs for varying external forces.
Resolving complex 3D geometries $\Rightarrow$ millions of degrees of freedom.

Analysis of elastic deformations requires many simulation runs for varying external forces.

Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) $\leadsto$ Craig-Bampton method — not discussed in this course!
A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where $x_{ij}$ contains color information of pixel $(i,j)$.

Memory: $4 \cdot n_x \cdot n_y$ bytes.
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**Theorem: (Schmidt-Mirsky/Eckart-Young)**

Best rank-$r$ approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\hat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U\Sigma V^T$ is the singular value decomposition (SVD) of $X$.

The approximation error is $\|X - \hat{X}\|_2 = \sigma_{r+1}$. 

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**Idea for dimension reduction**

Instead of $X$ save $u_1, \ldots, u_r$, $\sigma_1 v_1, \ldots, \sigma_r v_r$.

$\sim$ memory $= 4r \times (n_x + n_y)$ bytes.
Example: Image Compression by Truncated SVD

Example: Clown

320 \times 200 \text{ pixel} \\ \sim \approx 256 \text{ kb}
Example: Image Compression by Truncated SVD

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320 × 200 pixel

→ ≈ 256 kb

rank \( r = 50 \), ≈ 104 kb
Example: Image Compression by Truncated SVD

Example: Clown

- rank \( r = 50, \approx 104 \text{ kb} \)

- rank \( r = 20, \approx 42 \text{ kb} \)

320 \times 200 \text{ pixel} \quad \Rightarrow \quad \approx 256 \text{ kb}
Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:
James H. Wilkinson, Wallace Givens,
George Forsythe, Alston Householder,
Peter Henrici, Fritz L. Bauer.

640 × 480 pixel, ≈ 1229 kb
**Example: Gatlinburg**

Organizing committee
Gatlinburg/Householder Meeting 1964: 

- rank $r = 100$, $\approx 448$ kb
- rank $r = 50$, $\approx 224$ kb

640 $\times$ 480 pixel, $\approx 1229$ kb
Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices
Dynamical Systems

\[
\Sigma : \begin{cases}
\dot{x}(t) = f(t, x(t), u(t)), & x(t_0) = x_0, \\
y(t) = g(t, x(t), u(t))
\end{cases}
\]

with

- states \( x(t) \in \mathbb{R}^n \),
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Original System

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Goal:

\[ \|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \text{ for all admissible input signals.} \]
### Model Reduction for Dynamical Systems

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- states \(\hat{x}(t) \in \mathbb{R}^r, \ r \ll n\)
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**Secondary goal:** reconstruct approximation of \( x \) from \( \hat{x} \).
Linear, Time-Invariant (LTI) Systems

\[
\dot{x} = f(t, x, u) = Ax + Bu, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m},
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y = g(t, x, u) = Cx + Du, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}.
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Assumptions (for now): \( t_0 = 0, \quad x_0 = x(0) = 0, \quad D = 0. \)
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State-Space Description for I/O-Relation

Variation-of-constants \( \mapsto \)

\[ S : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)} Bu(\tau) \, d\tau \quad \text{for all } t \in \mathbb{R}. \]
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- Problem: in general, \( S \) does not have a discrete SVD and can therefore not be approximated as in the matrix case!
Model Reduction for Linear Systems

Linear, Time-Invariant (LTI) Systems

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Alternative to State-Space Operator: Hankel operator

Instead of

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use Hankel operator

\[ \mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau \quad \text{for all} \quad t > 0. \]
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\( \mathcal{H} \) compact, finite-dimensional \( \Rightarrow \) \( \mathcal{H} \) has discrete SVD

\( \sim \) Hankel singular values \( \{\sigma_j\}_{j=1}^{\infty} : \sigma_1 \geq \ldots \geq \sigma_n \geq \sigma_{n+1} = 0 = \ldots = 0. \)
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\(\sim\) Hankel singular values \(\{\sigma_j\}_{j=1}^{\infty} : \sigma_1 \geq \ldots \geq \sigma_n \geq \sigma_{n+1} = 0 = \ldots = 0.\)

\(\implies\) SVD-type approximation of \(\mathcal{H}\) possible!
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Hankel singular values

![Hankel Singular Values for Atmospheric Storm Model](chart.png)
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\( \Rightarrow \) solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

But: computationally infeasible for large-scale systems.
Linear, Time-Invariant (LTI) Systems

\[ \Sigma : \begin{cases} \dot{x} = Ax + Bu, & A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\ y = Cx + Du, & C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}. \end{cases} \]

Assumptions: \( t_0 = 0, \ x_0 = x(0) = 0. \)

Laplace Transform / Frequency Domain

Application of Laplace transform

\[ \mathcal{L} : x(t) \mapsto x(s) = \int_0^{\infty} e^{-st} x(t) \, dt \quad (\Rightarrow \dot{x}(t) \mapsto sx(s)) \]

with \( s \in \mathbb{C} \) leads to linear system of equations:

\[ sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s). \]
Linear, Time-Invariant (LTI) Systems

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\Sigma : \left\{ \begin{array}{c}
\dot{x} = Ax + Bu, \\
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\end{array} \right. \\
A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\
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Laplace Transform / Frequency Domain

\[
sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s)
\]
yields I/O-relation in frequency domain:

\[
y(s) = \left( C(sI_n - A)^{-1}B + D \right) u(s) = G(s)u(s).
\]

\( G \) is the transfer function of \( \Sigma \), \( G : \mathcal{L}_2^m \to \mathcal{L}_2^p \) \( (\mathcal{L}_2 := \mathcal{L}(\mathcal{L}_2(-\infty, \infty))) \).
Approximation Problem

Approximate the dynamical system

\[
\begin{align*}
\dot{x} &= Ax + Bu, & A &\in \mathbb{R}^{n\times n}, & B &\in \mathbb{R}^{n\times m}, \\
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\end{align*}
\]

by reduced-order system

\[
\begin{align*}
\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, & \hat{A} &\in \mathbb{R}^{r\times r}, & \hat{B} &\in \mathbb{R}^{r\times m}, \\
\hat{y} &= \hat{C}\hat{x} + \hat{D}u, & \hat{C} &\in \mathbb{R}^{p\times r}, & \hat{D} &\in \mathbb{R}^{p\times m}.
\end{align*}
\]

of order \(r \ll n\), such that

\[
\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \|u\| \leq \text{tolerance} \cdot \|u\|.
\]
Model Reduction as Approximation Problem

Approximation Problem

Approximate the dynamical system

\[
\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m},
\]

\[
y = Cx + Du, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}.
\]

by reduced-order system

\[
\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} \in \mathbb{R}^{r \times r}, \quad \hat{B} \in \mathbb{R}^{r \times m},
\]

\[
\hat{y} = \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{p \times r}, \quad \hat{D} \in \mathbb{R}^{p \times m}.
\]

of order \( r \ll n \), such that

\[
\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \|u\| \leq \text{tolerance} \cdot \|u\|.
\]

\[\implies\] Approximation problem: \( \min_{\text{order}(\hat{G}) \leq r} \|G - \hat{G}\| \).
A linear system

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \]

is **stable** if its transfer function \( G(s) \) has all its poles in the left half plane and it is **asymptotically** (or Lyapunov or exponentially) stable if all poles are in the open left half plane \( \mathbb{C}^- := \{ z \in \mathbb{C} | \Re(z) < 0 \} \).
Definition

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Lemma

Sufficient for asymptotic stability is that \( A \) is asymptotically stable (or Hurwitz), i.e., the spectrum of \( A \), denoted by \( \Lambda(A) \), satisfies \( \Lambda(A) \subset \mathbb{C}^- \).

Note that by abuse of notation, often stable system is used for asymptotically stable systems.
Questions:

- For fixed $x_0 \in \mathbb{R}^n$ and some $x^1 \in \mathbb{R}^n$, is there a feasible control function $u \in U_{ad}$ (e.g., $U_{ad} \in \{C^k[0, T], L_2(0, T), PC[0, T]\}$), possibly with constraints $u(t) \leq u(t) \leq \bar{u}(t)$) and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$? What is the set of targets $x^1$ reachable from $x^0$?
Questions:

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Note: for LTI systems $\dot{x} = Ax + Bu$, both concepts are equivalent!
Definition (Controllability)

Consider the target (the state to be reached) \( x^1 \in \mathbb{R}^n \).

a) An LTI system with initial value \( x(0) = x^0 \) is controllable to \( x^1 \) in time \( t_1 > 0 \) if there exists \( u \in U_{ad} \) such that \( x(t_1; u) = x^1 \).

(Equivalently, \( (t_1, x^1) \) is reachable from \( (0, x^0) \).)
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(Equivalently, $(t_1, x^1)$ is **reachable** from $(0, x^0)$.)

b) $x^0$ is **controllable to $x^1$** if there exists a $t_1 > 0$ such that $(t_1, x^1)$ can be reached from $(0, x^0)$.

c) If the system is controllable to $x^1$ for all $x^0 \in \mathbb{R}^n$, it is **(completely) controllable**.
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The controllability set w.r.t. $x^1$ is defined as $C := \bigcup_{t_1 > 0} C(t_1)$ where

$$C(t_1) := \{x^0 \in \mathbb{R}^n \mid \exists u \in U_{ad} : x(t_1; u) = x^1\}.$$
**Definition (Controllability)**

Consider the target (the state to be reached) $x^1 \in \mathbb{R}^n$.

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In short: an LTI system is controllable $\iff C = \mathbb{R}^n$. 
Now: characterize controllability.
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Variation of constants \( \Rightarrow \)

\[
x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}(x^0 + \int_0^t e^{-As}Bu(s)ds).
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Hence, if \( x^0 \) is controllable to \( x^1 \):

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x^1 = x(t_1) = e^{At_1}x^0 + \int_0^{t_1} e^{A(t_1-t)}Bu(t)dt
\]

This is equivalent to

\[
e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}Bu(t)dt.
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Ansatz: \( u(t) = B^T e^{-A^T}c \implies \)
\[
e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}BB^T e^{-A^T}dt c =: P(0, t_1)c.
\]
Now: characterize controllability.

Variation of constants \( \Rightarrow \)

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x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)\,ds = e^{At}(x^0 + \int_0^t e^{-As}Bu(s)\,ds).
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\]

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\[
e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At} BB^T e^{-A^T t} dt c =: P(0, t_1)c.
\]

Hence, an LTI system is controllable iff this linear system is solvable for \( c \in \mathbb{R}^n \), i.e., iff \( P(0, t_1) \) is invertible. (Note: \( P(0, t_1) = P(0, t_1)^T \geq 0 \) by definition!)
Now: characterize controllability.

**Theorem**

For an LTI system defined by $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$, T.F.A.E.:

a) The LTI system $\dot{x} = Ax + Bu$ is controllable.

b) The finite time Gramian $P(0, t_1)$ is spd $\forall t_1 > 0$.

c) The controllability matrix

$$K(A, B) := [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n \times n \cdot m}$$

has full rank $n$. *(Note: range$(K(A, B)) = C(t_1) \forall t_1 > 0!$)*

d) If $z$ is a left eigenvector of $A$, then $z^* B \neq 0$.

e) *(Hautus test)* $\text{rank}([\lambda I - A, B]) = n \ \forall \lambda \in \mathbb{C}$. 
The Gramian characterization of controllability for stable systems can be based on positive definiteness of the (infinite) controllability Gramian

\[ P := \int_{0}^{\infty} e^{As} BB^T e^{A^Ts} ds, \]

using congruence of \( P(0, t_1) \) to \( \int_{0}^{t_1} e^{As} BB^T e^{A^Ts} ds \) and taking the limit \( t_1 \to \infty \).
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**Theorem**

For a stable LTI system defined by \((A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\), T.F.A.E.:

a) The LTI system \( \dot{x} = Ax + Bu \) is controllable.

b) The controllability Gramian \( P \) is positive definite.
New question: suppose we have

\[ y(t) = \tilde{y}(t) \]

corresponding to two trajectories \( x, \tilde{x} \) obtained by the same input function \( u(t) \). Can we conclude that \( x(0) = \tilde{x}(0) \), or even stronger, that \( x(t) = \tilde{x}(t) \) for \( t \leq 0, t \geq 0 \) (past/future)?

(Note that \( x(t_0) = \tilde{x}(t_0) \) is sufficient as trajectory uniquely determined. In other words, is the mapping \( x^0 \rightarrow y(t) \) injective?)
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(Note that \( x(t_0) = \tilde{x}(t_0) \) is sufficient as trajectory uniquely determined. In other words, is the mapping \( x^0 \rightarrow y(t) \) injective?)

**Definition (Observability)**

An LTI system is **reconstructable (observable)** if for solution trajectories \( x(t), \tilde{x}(t) \) obtained with the same input function \( u \), we have

\[
\begin{align*}
y(t) &= \tilde{y}(t) \quad &\forall t \leq 0\quad (\forall t \geq 0) \\
\implies x(t) &= \tilde{x}(t) \quad &\forall t \leq 0\quad (\forall t \geq 0).
\end{align*}
\]
Characterization of observability/reconstructability:

**Theorem (Duality)**

An LTI system is reconstructable if and only if the dual system
\[ \dot{x}(t) = -A^T x(t) - C^T u(t) \]
is controllable.
Theorem (Duality)

An LTI system is reconstructable if and only if the dual system
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Theorem

For an LTI system defined by \((A, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{p \times n}\), T.F.A.E.:

a) The LTI system is reconstructable.

b) The LTI system is observable.

c) The observability matrix
\[ O(A, C) = \begin{bmatrix} C^T, A^T C^T, (A^2)^T C, \ldots, (A^{n-1})^T C^T \end{bmatrix}^T \in \mathbb{R}^{np \times n} \text{ has rank } n. \]

d) If \(Ax = \lambda x\), then \(C^T x \neq 0\).

e) (Hautus test) \(\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n\).
Characterization of observability/reconstructability:

**Theorem (Duality)**

An LTI system is reconstructable if and only if the dual system
\[
\dot{x}(t) = -A^T x(t) - C^T u(t)
\]
is controllable.

**Theorem**

A stable LTI system is observable if and only if the observability Gramian
\[
Q := \int_{0}^{\infty} e^{A^T t} C^T Ce^{At} dt
\]
is symmetric positive definite.
Controllability/observability are sometimes too strong.
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Weaker requirement: is there $u \in U_{ad}$ to steer $x_0$ to vicinity of $x^1$?
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For LTI systems, it suffices to consider \( x^1 = 0! \).
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- For LTI systems, it suffices to consider \( x^1 = 0 \!\!1 \)
- Hence, is there \( u \in \mathcal{U}_{ad} \) so that \( \lim_{t \to \infty} x(t; u) = 0 \) (\( \forall x^0 \in \mathbb{R}^n \))?
Controllability/observability are sometimes too strong.

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Hence, is there \( u \in \mathcal{U}_{ad} \) so that \( \lim_{t \to \infty} x(t; u) = 0 \) (\( \forall x^0 \in \mathbb{R}^n \))?

If the answer is **yes**, then the LTI system is called **stabilizable**
Controllability/observability are sometimes too strong.

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**Theorem**

For an LTI system defined by \((A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\), T.F.A.E.:

a) The LTI system is stabilizable.

b) \( \exists \) feedback operator/matrix \( F \in \mathbb{R}^{m \times n} \) with \( \Lambda(A + BF) \subset \mathbb{C}^- \).

c) If \( p^* A = \tilde{\lambda} p^* \) and \( \text{Re}(\lambda) \geq 0 \), then \( p^* B \neq 0 \).

d) \( \text{rank}([A - \lambda I, B]) = n\) \( \forall \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) \geq 0 \).

e) \( \Lambda(A_3) \subset \mathbb{C}^- \) in the \textbf{(controllability) Kalman decomposition} of \((A, B)\),

\[
V^T AV = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad V^T B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.
\]
∃ dual concept of stabilizability, analogous to duality of controllability and observability.

**Definition (Detectability)**

An LTI system is detectable if for any solution $x(t)$ of $\dot{x} = Ax$ with $Cx(t) \equiv 0$ we have $\lim_{t \to \infty} x(t) = 0$.

(We can not observe all of $x$, but the unobservable part is stable.)
∃ dual concept of stabilizability, analogous to duality of controllability and observability.

**Theorem**

For an LTI system defined by \((A, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{p \times n}\), T.F.A.E.:

a) The LTI system is detectable.

b) \((A^T, C^T)\) is stabilizable.

c) \(Ax = \lambda x, \text{Re}(\lambda) \geq 0 \Rightarrow C^T x \neq 0\).

d) \(\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n\) for all \(\lambda, \text{Re}(\lambda) \geq 0\).

e) In the observability Kalman decomposition of \((A^T, C^T)\),

\[ W^T AW = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, CW = [C_1 0], \]

we have \(\Lambda(A_3) \subset \mathbb{C}^-\).
Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with transfer function} \\ y(t) = Cx(t) + Du(t), & G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple \((A, B, C, D)\in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}\) is called a realization of \(\Sigma\).
Realization Theory for Linear Systems
Basic principles

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Realizations are not unique!
Transfer function is invariant under state-space transformations,

\[ \mathcal{T} : \begin{cases} x \rightarrow T x, \\ (A, B, C, D) \rightarrow (TAT^{-1}, TB, CT^{-1}, D), \end{cases} \]
Realization Theory for Linear Systems

Basic principles

**Definition**

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\Sigma: \begin{cases}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{cases}
\]

with transfer function

\[
G(s) = C(sI - A)^{-1}B + D,
\]

the quadruple \((A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}\) is called a realization of \(\Sigma\).

**Realizations are not unique!**

Transfer function is invariant under addition of uncontrollable/unobservable states:

\[
\frac{d}{dt} \begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t),
\]

\[
\frac{d}{dt} \begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix} \begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t),
\]

for arbitrary \(A_j \in \mathbb{R}^{n_j \times n_j}, \ j = 1, 2, \ B_1 \in \mathbb{R}^{n_1 \times m}, \ C_2 \in \mathbb{R}^{q \times n_2}\) and any \(n_1, n_2 \in \mathbb{N}\).
### Definition

For a linear (time-invariant) system

\[ \Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with transfer function} \\ y(t) = Cx(t) + Du(t), & G(s) = C(sI - A)^{-1}B + D, \end{cases} \]

the quadruple \((A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}\) is called a realization of \(\Sigma\).

### Realizations are not unique!

Hence,

\[
(A, B, C, D), \quad \left( \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),
\]

\[
(TAT^{-1}, TB, CT^{-1}, D), \quad \left( \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),
\]

are all realizations of \(\Sigma\)!
### Definition

For a linear (time-invariant) system

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\dot{x}(t) = Ax(t) + Bu(t), \\
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G(s) = C(sI - A)^{-1}B + D,
\]

the quadruple \((A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}\) is called a realization of \(\Sigma\).

### Definition

The **McMillan degree** of \(\Sigma\) is the unique minimal number \(\hat{n} \geq 0\) of states necessary to describe the input-output behavior completely.

A **minimal realization** is a realization \((\hat{A}, \hat{B}, \hat{C}, \hat{D})\) of \(\Sigma\) with order \(\hat{n}\).
Definition

For a linear (time-invariant) system

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\Sigma : \begin{cases}
\dot{x}(t) = Ax(t) + Bu(t), \\
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\end{cases}
\]

with transfer function

\[G(s) = C(sI - A)^{-1}B + D,\]

the quadruple \((A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}\) is called a realization of \(\Sigma\).

Definition

The McMillan degree of \(\Sigma\) is the unique minimal number \(\hat{n} \geq 0\) of states necessary to describe the input-output behavior completely. A minimal realization is a realization \((\hat{A}, \hat{B}, \hat{C}, \hat{D})\) of \(\Sigma\) with order \(\hat{n}\).

Theorem

A realization \((A, B, C, D)\) of a linear system is minimal \(\iff (A, B)\) is controllable and \((A, C)\) is observable.
<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A realization ((A, B, C, D)) of a linear system (\Sigma) is <strong>balanced</strong> if its infinite controllability/observability Gramians (P/Q) satisfy</td>
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<td>[ P = Q = \text{diag}{\sigma_1, \ldots, \sigma_n} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1}, \ j = 1, \ldots, n-1). ]</td>
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When does a balanced realization exist?
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\]

When does a balanced realization exist?
Assume \(A\) to be Hurwitz, i.e. \(\Lambda(A) \subset \mathbb{C}^-\). Then:

Theorem

Given a stable minimal linear system \(\Sigma : (A, B, C, D)\), a balanced realization is obtained by the state-space transformation with

\[
T_b := \Sigma^{-\frac{1}{2}} V^T R,
\]

where \(P = S^T S\), \(Q = R^T R\) (e.g., Cholesky decompositions) and \(SR^T = U\Sigma V^T\) is the SVD of \(SR^T\).
A realization \((A, B, C, D)\) of a **stable** linear system \(\Sigma\) is **balanced** if its infinite controllability/observability Gramians \(P/Q\) satisfy

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\(\sigma_1, \ldots, \sigma_n\) are the **Hankel singular values** of \(\Sigma\).

**Note:** \(\sigma_1, \ldots, \sigma_n \geq 0\) as \(P, Q \geq 0\) by definition, and \(\sigma_1, \ldots, \sigma_n > 0\) in case of minimality!
Realization Theory for Linear Systems

Balanced Realizations

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Theorem

The Hankel singular values (HSV) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!
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Proof. In balanced coordinates, the HSVs are \( \Lambda (PQ)^{\frac{1}{2}} \). Now let

\[
(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)
\]

be any transformed realization with associated controllability Lyapunov equation

\[
0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.
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The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

**Proof.** In balanced coordinates, the HSVs are $\Lambda \left( PQ \right)^{\frac{1}{2}}$. Now let

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The uniqueness of the solution of the Lyapunov equation implies that $\hat{P} = TPT^T$ and, analogously, $\hat{Q} = T^{-T}QT^{-1}$. Therefore,

$$\hat{P}\hat{Q} = TPQT^{-1},$$

showing that $\Lambda \left( \hat{P}\hat{Q} \right) = \Lambda \left( PQ \right) = \{\sigma_1^2, \ldots, \sigma_n^2\}$. 
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A realization \((A, B, C, D)\) of a stable linear system \(\Sigma\) is balanced if its infinite controllability/observability Gramians \(P/Q\) satisfy

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Note: \(\sigma_1, \ldots, \sigma_n \geq 0\) as \(P, Q \geq 0\) by definition, and \(\sigma_1, \ldots, \sigma_n > 0\) in case of minimality!

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading \(\hat{n} \times \hat{n}\) submatrices equal to \(\text{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})\), and

\[ \hat{P} \hat{Q} = \text{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0). \]

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].
Consider the transfer function

\[ G(s) = C (sl - A)^{-1} B + D \]

and input functions \( u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty) \), with the 2-norm

\[
\|u\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u^*(j\omega)u(j\omega) \, d\omega.
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Assume \( A \) is (asymptotically) stable: \( \Lambda(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} : \text{re} \, z < 0\} \).
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Then for all \( s \in C^+ \cup j\mathbb{R} \), \( \|G(s)\| \leq M \leq \infty \Rightarrow \)

\[ \int_{-\infty}^{\infty} y^*(j\omega) y(j\omega) \, d\omega = \int_{-\infty}^{\infty} u^*(j\omega) G^*(j\omega) G(j\omega) u(j\omega) \, d\omega \]
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(Here:, \( \| \cdot \| \) denotes the Euclidian vector or spectral matrix norm.)
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\[ \implies y \in L^p_2(-\infty, \infty) \cong L^p_2. \]
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Assume \( A \) is (asymptotically) stable: \( \Lambda(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} : \text{re } z < 0\} \). Consequently, the 2-induced operator norm

\[ \|G\|_\infty := \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2} \]

is well defined. It can be shown that

\[ \|G\|_\infty := \sup_{\omega \in \mathbb{R}} \|G(j\omega)\| = \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(G(j\omega)). \]
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**Hardy space \( \mathcal{H}_\infty \)**

Function space of analytic and bounded (in \( \mathbb{C}^+ \)) matrix-/scalar-valued functions. The \( \mathcal{H}_\infty \)-norm is

\[ \|F\|_\infty := \sup_{\text{re} \, s > 0} \sigma_{\text{max}}(F(s)) = \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(F(j\omega)). \]

Stable transfer functions are in the Hardy spaces

- \( \mathcal{H}_\infty \) in the SISO case (single-input, single-output, \( m = p = 1 \));
- \( \mathcal{H}^p \times m_\infty \) in the MIMO case (multi-input, multi-output, \( m > 1, p > 1 \)).
Consider the transfer function

\[ G(s) = C \left(s I - A\right)^{-1} B + D. \]

**Paley-Wiener Theorem (Parseval’s equation/Plancherel Theorem)**

\[ L_2(-\infty, \infty) \cong L_2, \quad L_2(0, \infty) \cong H_2 \]

Consequently, 2-norms in time and frequency domains coincide!
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**H∞ approximation error**

Reduced-order model \( \Rightarrow \) transfer function \( \hat{G}(s) = \hat{C}(sl_r - \hat{A})^{-1}\hat{B} + \hat{D}. \)

\[ \| y - \hat{y} \|_2 = \| Gu - \hat{G} u \|_2 \leq \| G - \hat{G} \|_\infty \| u \|_2. \]

\( \implies \) compute reduced-order model such that \( \| G - \hat{G} \|_\infty < tol! \)

Note: error bound holds in time- and frequency domain due to Paley-Wiener!
Consider transfer function \( G(s) = C (sI - A)^{-1} B \), i.e. \( D = 0 \).

**Hardy space \( \mathcal{H}_2 \)**

Function space of matrix-/scalar-valued functions that are analytic in \( \mathbb{C}^+ \) and bounded w.r.t. the \( \mathcal{H}_2 \)-norm

\[
\| F \|_2 := \left( \sup_{\text{Re} \sigma > 0} \int_{-\infty}^{\infty} \| F(\sigma + j\omega) \|^2_F \, d\omega \right)^{\frac{1}{2}} \\
= \left( \int_{-\infty}^{\infty} \| F(j\omega) \|^2_F \, d\omega \right)^{\frac{1}{2}}.
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- \( \mathcal{H}_2 \) in the SISO case (single-input, single-output, \( m = p = 1 \));
- \( \mathcal{H}_2^{p \times m} \) in the MIMO case (multi-input, multi-output, \( m > 1, p > 1 \)).
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\| F \|_2 = \left( \int_{-\infty}^{\infty} \| F(j\omega) \|_F^2 \ d\omega \right)^{\frac{1}{2}}.
\]

**\( \mathcal{H}_2 \) approximation error for impulse response \( (u(t) = u_0 \delta(t)) \)**

Reduced-order model \( \Rightarrow \) transfer function \( \hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1} \hat{B} \).

\[
\| y - \hat{y} \|_2 = \| Gu_0 \delta - \hat{G} u_0 \delta \|_2 \leq \| G - \hat{G} \|_2 \| u_0 \|.
\]

\( \implies \) compute reduced-order model such that \( \| G - \hat{G} \|_2 < \text{tol}! \)
### Qualitative and Quantitative Study of the Approximation Error

**Approximation Problems**

<table>
<thead>
<tr>
<th>Norm</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H}_\infty$-norm</td>
<td>best approximation problem for given reduced order $r$ in general open; <strong>balanced truncation</strong> yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.</td>
</tr>
<tr>
<td>$\mathcal{H}_2$-norm</td>
<td>necessary conditions for best approximation known; (local) optimizer computable with <strong>iterative rational Krylov algorithm (IRKA)</strong></td>
</tr>
<tr>
<td>Hankel-norm $|G|<em>H := \sigma</em>{\text{max}}$</td>
<td>optimal Hankel norm approximation (AAK theory)</td>
</tr>
</tbody>
</table>

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Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- **absolute errors** \( \| G(j\omega_j) - \hat{G}(j\omega_j) \|_2, \| G(j\omega_j) - \hat{G}(j\omega_j) \|_\infty (j = 1, \ldots, N_\omega); \)
- **relative errors** \( \frac{\| G(j\omega_j) - \hat{G}(j\omega_j) \|_2}{\| G(j\omega_j) \|_2}, \frac{\| G(j\omega_j) - \hat{G}(j\omega_j) \|_\infty}{\| G(j\omega_j) \|_\infty}; \)
- **"eyeball norm", i.e. look at frequency response/Bode (magnitude) plot:**
  - for SISO system, log-log plot frequency vs. \( |G(j\omega)| \) (or \( |G(j\omega) - \hat{G}(j\omega)| \)) in decibels, \( 1 \text{ dB} \approx 20 \log_{10}(\text{value}) \);
  - for MIMO systems, \( p \times m \) array of plots \( G_{ij} \).
1. Introduction

2. Model Reduction by Projection
   - Projection Basics
   - Extensions

3. Balanced Truncation

4. Final Remarks
Automatic generation of compact models.
Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

\[ \|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m). \]

\[ \implies \text{Need computable error bound/estimate!} \]
Automatic generation of compact models.

Satisfy desired error tolerance for all admissible input signals, i.e., want

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Model Reduction by Projection

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- Preserve physical properties:
  - stability (poles of \( G \) in \( \mathbb{C}^- \)),
  - minimum phase (zeroes of \( G \) in \( \mathbb{C}^- \)),

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Balancing-based Methods
Automatic generation of compact models.

Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\| y - \hat{y} \| < \text{tolerance} \cdot \| u \| \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$$\implies \text{Need computable error bound/estimate!}$$

Preserve physical properties:

- stability (poles of $G$ in $\mathbb{C}^-$),
- minimum phase (zeroes of $G$ in $\mathbb{C}^-$),
- passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau) \, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

(“system does not generate energy”).
A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \text{range}(P)$, then $P$ is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $\mathcal{V} = [v_1, \ldots, v_r]$, then $P = V(V^T V)^{-1}V^T$ is a projector onto $\mathcal{V}$. 
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Properties:

- If $P = P^T$, then $P$ is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector. (aka: Petrov-Galerkin projection.)
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- $P$ is the identity operator on $\mathcal{V}$, i.e., $Pv = v \ \forall v \in \mathcal{V}$. 

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- $P$ is the identity operator on $\mathcal{V}$, i.e., $Pv = v \ \forall v \in \mathcal{V}$.

- $I - P$ is the complementary projector onto ker $P$. 
A projector is a matrix \( P \in \mathbb{R}^{n \times n} \) with \( P^2 = P \). Let \( \mathcal{V} = \text{range}(P) \), then \( P \) is a projector onto \( \mathcal{V} \). On the other hand, if \( \{v_1, \ldots, v_r\} \) is a basis of \( \mathcal{V} \) and \( \mathcal{V} = [v_1, \ldots, v_r] \), then \( P = \mathcal{V}(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \) is a projector onto \( \mathcal{V} \).

**Properties:**

- If \( P = P^T \), then \( P \) is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector. (aka: Petrov-Galerkin projection.)
- \( P \) is the identity operator on \( \mathcal{V} \), i.e., \( P \mathbf{v} = \mathbf{v} \ \forall \mathbf{v} \in \mathcal{V} \).
- \( I - P \) is the complementary projector onto \( \text{ker} \ P \).
- If \( \mathcal{V} \) is an \( A \)-invariant subspace corresponding to a subset of \( A \)'s spectrum, then we call \( P \) a spectral projector.
A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \text{range}(P)$, then $P$ is projector onto $\mathcal{V}$. On the other hand, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a basis of $\mathcal{V}$ and $\mathcal{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_r]$, then $P = \mathcal{V}(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T$ is a projector onto $\mathcal{V}$.

Properties:

- If $P = P^T$, then $P$ is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector. (aka: Petrov-Galerkin projection.)
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- $I - P$ is the complementary projector onto ker $P$.
- If $\mathcal{V}$ is an $A$-invariant subspace corresponding to a subset of $A$'s spectrum, then we call $P$ a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another $r$-dimensional subspace and $\mathcal{W} = [w_1, \ldots, w_r]$ be a basis matrix for $\mathcal{W}$, then $P = \mathcal{V}(\mathcal{W}^T \mathcal{V})^{-1} \mathcal{W}^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.
Model Reduction by Projection

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods) → Part II of tutorial, by Serkan Gugercin!
3. Balanced Truncation
4. many more...

Joint feature of these methods: computation of reduced-order model (ROM) by projection!
Joint feature of these methods: computation of reduced-order model (ROM) by projection!
Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx \mathcal{V}W^Tx =: \hat{x}$, where

$$\text{range}(\mathcal{V}) = \mathcal{V}, \quad \text{range}(\mathcal{W}) = \mathcal{W}, \quad W^TV = I_r.$$ 

Then, with $\hat{x} = W^Tx$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \hat{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^TAV, \quad \hat{B} := W^TB, \quad \hat{C} := CV, \quad (\hat{D} := D).$$
Joint feature of these methods: computation of reduced-order model (ROM) by projection!
Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^Tx =: \tilde{x}$, and the reduced-order model is $\tilde{x} = W^T x$

$$
\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).
$$

Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$
W^T (\dot{\tilde{x}} - A\tilde{x} - Bu) = W^T (VW^T \dot{x} - AVW^T x - Bu)
$$
Joint feature of these methods: computation of reduced-order model (ROM) by projection!
Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, and the reduced-order model is

$$\dot{\tilde{x}} = W^T x$$

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Joint feature of these methods: computation of reduced-order model (ROM) by projection!

Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx \mathcal{V}W^Tx =: \hat{x}$, and the reduced-order model is $\hat{x} = W^Tx$

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Important observation:

- The state equation residual satisfies $\dot{\hat{x}} - \hat{A}\hat{x} - \hat{B}u \perp \mathcal{W}$, since

$$W^T(\dot{\hat{x}} - \hat{A}\hat{x} - \hat{B}u) = W^T(VW^T\dot{x} - AVW^Tx - Bu) = W^T\dot{x} - W^TAVW^Tx - W^TBu = \dot{\hat{x}} - \hat{A}\hat{x} - \hat{B}u = 0.$$
Given the ROM

\[
\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),
\]

the error transfer function can be written as

\[
G(s) - \hat{G}(s) = (C(sI_n - A)^{-1}B + D) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right)
\]
Given the ROM

\[ \hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D), \]

the error transfer function can be written as

\[
G(s) - \hat{G}(s) = \left( C(sI_n - A)^{-1}B + D \right) - \left( \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D} \right)
\]

\[
= C \left( (sI_n - A)^{-1} - V(sI_r - \hat{A})^{-1}W^T \right) B
\]
Projection \sim Rational Interpolation

Given the ROM

\[ \hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D), \]

the error transfer function can be written as

\[
G(s) - \hat{G}(s) = \left( C(sI_n - A)^{-1} B + D \right) - \left( \hat{C}(sI_n - \hat{A})^{-1} \hat{B} + \hat{D} \right) \\
= C \left( (sI_n - A)^{-1} - V(sI_r - \hat{A})^{-1} W^T \right) B \\
= C \left( I_n - V(sI_r - \hat{A})^{-1} W^T (sI_n - A) \right) (sI_n - A)^{-1} B.
\]

\[=: P(s) \]
Projection $\sim$ Rational Interpolation

Given the ROM
\[ \hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D), \]
the error transfer function can be written as
\[
G(s) - \hat{G}(s) = (C(sI_n - A)^{-1}B + D) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right)
\]
\[
= C\left(I_n - V(sI_r - \hat{A})^{-1}W^T(sI_n - A)\right)(sI_n - A)^{-1}B.
\]

\[ P(s) \] is a projector onto \( \mathcal{V} \):
\[
\text{range}(P(s)) \subset \text{range}(V), \text{ all matrices have full rank } \Rightarrow "\Rightarrow " \text{, and }
\]
\[
P(s)^2 = V(sI_r - \hat{A})^{-1}W^T(sI_n - A)V(sI_r - \hat{A})^{-1}W^T(sI_n - A)
\]
Projection $\leadsto$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) = (C(sI_n - A)^{-1}B + D) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right)$$

$$= C\left(I_n - V(sI_r - \hat{A})^{-1}W^T(sI_n - A)\right)(sI_n - A)^{-1}B.$$

$P(s)$ is a projector onto $\mathcal{V}$:

$$\text{range}(P(s)) \subset \text{range}(V), \text{ all matrices have full rank} \Rightarrow "=", \text{ and}$$

$$P(s)^2 = V(sI_r - \hat{A})^{-1}W^T(sI_n - A)V(sI_r - \hat{A})^{-1}W^T(sI_n - A)$$

$$= V(sI_r - \hat{A})^{-1}(sI_r - \hat{A})(sI_r - \hat{A})^{-1}W^T(sI_n - A) = P(s).$$
**Projection ⇝ Rational Interpolation**

Given the ROM

\[
\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),
\]

the error transfer function can be written as

\[
G(s) - \hat{G}(s) = (C(sl_n - A)^{-1}B + D) - \left(\hat{C}(sl_n - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\
= C \left(I_n - V(sl_r - \hat{A})^{-1}W^T(sl_n - A)\right)(sl_n - A)^{-1}B.
\]

\[P(s)\] is a projector onto \(V\) \(\implies\)

Given \(s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A})\right)\),

if \((s_* l_n - A)^{-1}B \in V\), then \((I_n - P(s_*))(s_* l_n - A)^{-1}B = 0)\,

hence \(G(s_*) - \hat{G}(s_*) = 0 \implies G(s_*) = \hat{G}(s_*)\), i.e., \(\hat{G}\) interpolates \(G\) in \(s_*\)!
Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) = (C(sl_n - A)^{-1}B + D) - \left(\hat{C}(sl_n - \hat{A})^{-1}\hat{B} + \hat{D}\right)$$

$$= C\left(I_n - V(sl_r - \hat{A})^{-1}W^T(sl_n - A)\right)(sl_n - A)^{-1}B.$$  

$$= : P(s)$$

Analogously,

$$Q(s)^*$$ is a projector onto $\mathcal{W}$ $\implies$ Given $s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A})\right)$,

$$if \ (s_* I_n - A)^{-T} C^T \in \mathcal{W}, then \ C(s_* I_n - A)^{-1}(I_n - Q(s_*)) = 0,$$

hence $G(s_*) - \hat{G}(s_*) = 0 \implies G(s_*) = \hat{G}(s_*$), i.e., $\hat{G}$ interpolates $G$ in $s_*$. 
Theorem \[[\text{Grimme 1997, Villemagne/Skelton 1987}]\]

Given the ROM

\[
\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),
\]

and \( s_* \in \mathbb{C} \setminus (\Lambda (A) \cup \Lambda (\hat{A})) \), if either

- \( (s_* I_n - A)^{-1} B \in \text{range}(V) \), or
- \( (s_* I_n - A)^{-T} C^T \in \text{range}(W) \),

then at \( s = s_* \), we obtain the (rational) interpolation condition

\[
G(s_*) = \hat{G}(s_*).
\]

Note: extension to Hermite interpolation \( \sim \) Part II!
**Base enrichment**

**Static modes** are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \implies x(t) \equiv -A^{-1}b_j.$$

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[b_1, \ldots, b_m] = A^{-1}B$.

**Interpolation-projection framework** $\implies G(0) = \hat{G}(0)!$

If two-sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \implies G'(0) = \hat{G}'(0)!$

Note: if $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$. 
Guyan reduction (static condensation)

Partition states in masters $x_1 \in \mathbb{R}^r$ and slaves $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)

Assume stationarity, i.e., $\dot{x} = 0$ and solve for $x_2$ in

$$0 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$\Rightarrow x_2 = -A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}B_2 u.$$

Inserting this into the first part of the dynamic system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u, \quad y = C_1x_1 + C_2x_2$$

then yields the reduced-order model

$$\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2) u$$

$$y = (C_1 - C_2A_{22}^{-1}A_{21})x_1 - C_2A_{22}^{-1}B_2 u.$$
1. Introduction

2. Model Reduction by Projection

3. Balanced Truncation
   - The basic method
   - ADI Methods for Lyapunov Equations
   - Balancing-Related Model Reduction

4. Final Remarks
Balanced Truncation

Basic principle:

Recall: an LTI system $\Sigma$, realized by $(A, B, C, D)$, is called balanced, if the Gramians, i.e., solutions $P, Q$ of the Lyapunov equations

\[
AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0,
\]
satisfy: $P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$. 

$\Lambda(PQ)_{1/2}$ are the Hankel singular values (HSVs) of $\Sigma$. 
Recall: an LTI system $\Sigma$, realized by $(A, B, C, D)$, is called balanced, if the Gramians, i.e., solutions $P, Q$ of the Lyapunov equations

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satisfy: $P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

$\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV) of $\Sigma$. 
### Basic principle:

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \)
- \( \Lambda (PQ)^{1/2} = \{ \sigma_1, \ldots, \sigma_n \} \) are the Hankel singular values (HSV) of \( \Sigma \).

**Proof:** Recall Hankel operator

\[
y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau
\]
Balanced Truncation

**Basic principle:**

- Lyapunov eqns.: $AP + PA^T + BB^T = 0$, $A^T Q + QA + C^T C = 0$.
- $\Lambda (PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV) of $\Sigma$.

**Proof:** Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau =: Ce^{At} \int_{-\infty}^{0} e^{-A\tau} Bu(\tau) \, d\tau =: z$$
### Basic principle:

- **Lyapunov eqns.**:
  \[ AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \]

- \[ \Lambda \left( PQ \right)^{\frac{1}{2}} = \{ \sigma_1, \ldots, \sigma_n \} \] are the **Hankel singular values (HSV)**s of \( \Sigma \).

**Proof:**

Recall Hankel operator

\[
y(t) = H u(t) = \int_{-\infty}^{0} C e^{A(t-\tau)} B u(\tau) \, d\tau =: C e^{At} \int_{-\infty}^{0} e^{-A\tau} B u(\tau) \, d\tau = C e^{At} z.
\]
**Basic principle:**

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0$.
- $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV) of $\Sigma$.

**Proof:** Recall Hankel operator

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Hankel singular values $=$ square roots of eigenvalues of $H^*H$. 

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Balancing-based Methods
Balanced Truncation

**Basic principle:**

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0 \).
- \( \Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\} \) are the Hankel singular values (HSV) of \( \Sigma \).

**Proof:** Recall Hankel operator

\[
y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau = Ce^{At} z.
\]

Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^* \mathcal{H} \),

\[
\mathcal{H}^* y(t) = \int_{0}^{\infty} B^T e^{A^T(\tau-t)} C^T y(\tau) \, d\tau
\]
Balanced Truncation

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0$, $A^T Q + QA + C^T C = 0$.
- $\Lambda (PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV$s$) of $\Sigma$.

Proof: Recall Hankel operator

$$y(t) = \mathcal{H} u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau = Ce^{At} z.$$ 

Hankel singular values = square roots of eigenvalues of $\mathcal{H}^* \mathcal{H}$,

$$\mathcal{H}^* y(t) = \int_{0}^{\infty} B^T e^{A^T(t-\tau)} C^T y(\tau) \, d\tau = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T y(\tau) \, d\tau.$$ 

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Balancing-based Methods 36/52
**Basic principle:**

- **Lyapunov eqns.:** \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \)
- \( \Lambda \left( PQ \right)^{1/2} = \{ \sigma_1, \ldots, \sigma_n \} \) are the Hankel singular values (HSV) of \( \Sigma \).

**Proof:** Recall Hankel operator

\[
y(t) = \mathcal{H} u(t) = \int_{-\infty}^{0} C e^{A(t-\tau)} B u(\tau) \, d\tau = C e^{At} z.
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Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^* \mathcal{H} \),

\[
\mathcal{H}^* y(t) = = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T y(\tau) \, d\tau.
\]

Hence,

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T C e^{A\tau} z \, d\tau
\]
Balanced Truncation

**Basic principle:**

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0 \).
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\]

Hence,

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T Ce^{A^T} z \, d\tau
\]

\[
= B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T Ce^{A^T} \, d\tau \, z
\]

\[
\equiv Q
\]
Balanced Truncation

Basic principle:

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \)
- \( \Lambda(PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\} \) are the Hankel singular values (HSV) of \( \Sigma \).

Proof: Recall Hankel operator

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y(t) = \mathcal{H} u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) d\tau = Ce^{At} z.
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\[
\mathcal{H}^* y(t) = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T y(\tau) d\tau.
\]

Hence,

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} \int_{0}^{\infty} e^{A^T \tau} C^T Ce^{A \tau} z d\tau
\]

\[= B^T e^{-A^T t} Q z \]
Balanced Truncation

Basic principle:

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \)

- \( \Lambda (PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\} \) are the Hankel singular values (HSV) of \( \Sigma \).

**Proof:**

Recall Hankel operator

\[
y(t) = \mathcal{H} u(t) = \int_{-\infty}^{0} C e^{A(t-\tau)} Bu(\tau) \, d\tau = Ce^{At} z.
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\]

Hence,

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} Qz
\]
**Balanced Truncation**

**Basic principle:**

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0$.
- $\Lambda (PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV) of $\Sigma$.

**Proof:** Recall Hankel operator

\[
y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau) \, d\tau = Ce^{At} z.
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Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

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\]

Hence,

\[
\mathcal{H}^*\mathcal{H}u(t) = B^T e^{-A^T t} Qz = \sigma^2 u(t).
\]
Balanced Truncation

Basic principle:

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0 \).

- \( \Lambda(PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\} \) are the Hankel singular values (HSV) of \( \Sigma \).

**Proof:** Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^* \mathcal{H} \),

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} Qz = \sigma^2 u(t).
\]

\[\implies u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz\]
**Basic principle:**

- Lyapunov eqns.: \( AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \)
- \( \Lambda (PQ)^{1/2} = \{\sigma_1, \ldots, \sigma_n\} \) are the Hankel singular values (HSV) of \( \Sigma \).

**Proof:** Hankel singular values = square roots of eigenvalues of \( \mathcal{H}^* \mathcal{H} \),

\[
\mathcal{H}^* \mathcal{H} u(t) = B^T e^{-A^T t} Q z = \sigma^2 u(t).
\]

\[
\implies u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Q z \implies \text{(recalling } z = \int_{-\infty}^{0} e^{-A \tau} Bu(\tau) d\tau)\]
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\[
z = \int_{-\infty}^{0} e^{-A\tau} B \frac{1}{\sigma^2} B^T e^{-A^T \tau} Q z \ d\tau
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Balanced Truncation

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\[\Rightarrow u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Q z \Rightarrow (\text{recalling } z = \int_0^\infty e^{-A\tau} B u(\tau) d\tau)\]

\[
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\[= \frac{1}{\sigma^2} \int_{-\infty}^0 e^{-A\tau} BB^T e^{-A^T \tau} d\tau Q z\]
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\[
H^*H u(t) = B^T e^{-A^T t} Q z \overset{=}\sim \sigma^2 u(t).
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\]

\[
z = \int_{-\infty}^0 e^{-A \tau} B \frac{1}{\sigma^2} B^T e^{-A^T \tau} Q z d\tau = \frac{1}{\sigma^2} \int_{-\infty}^0 e^{-A \tau} B B^T e^{-A^T \tau} d\tau Q z
\]

\[
= \frac{1}{\sigma^2} \int_{-\infty}^\infty e^{A t} B B^T e^{A^T t} dt Qz
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\[\iff PQz = \sigma^2 z.\]
**Basic principle:**

- Recall: an LTI system $\Sigma$, realized by $(A, B, C, D)$, is called balanced, if the Gramians, i.e., solutions $P, Q$ of the Lyapunov equations
  
  $$AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0,$$

  satisfy: $P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV$s$) of $\Sigma$.

- Compute balanced realization of the system via state-space transformation

$$\mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$$

  $$= \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right)$$
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  \end{bmatrix}, \begin{bmatrix}
  B_1 \\
  B_2
  \end{bmatrix}, \begin{bmatrix}
  C_1 & C_2
  \end{bmatrix}, D \right)
  \]
- Truncation $\leadsto (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$. 
Motivation:

HSV\text{s} are \textbf{system invariants}: they are preserved under $T: (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$: 

in transformed coordinates, the Gramians satisfy

\[
(TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T = 0,
\]
\[
(TAT^{-1})^T(TT^TQT^{-1}) + (TT^TQT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) = 0
\]

\[
\Rightarrow (TPT^T)(TT^TQT^{-1}) = TPQT^{-1},
\]

hence $\Lambda(PQ) = \Lambda((TPT^T)(TT^TQT^{-1}))$. 
Motivation:

HSV s are system invariants: they are preserved under
\( \mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D) \).

HSV s determine the energy transfer given by the Hankel map

\[ \mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+ . \]

In balanced coordinates ... energy transfer from \( u_- \) to \( y_+ \):

\[
E := \sup_{\substack{u \in L_2(-\infty, 0] \\ x(0) = x_0}} \int_0^\infty y(t)^T y(t) \, dt \\
\int_{-\infty}^0 u(t)^T u(t) \, dt = \frac{1}{\|x_0\|_2^n} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2
\]
Motivation:

HSV are system invariants: they are preserved under
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E := \sup_{u \in L_2(-\infty, 0)} \frac{\int_0^\infty y(t)^T y(t) \, dt}{\int_{-\infty}^0 u(t)^T u(t) \, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2
\]

\( \implies \) Truncate states corresponding to “small” HSVs
\( \implies \) complete analogy to best approximation via SVD!
### Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, \( P = S^T S \), \( Q = R^T R \).
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3. ROM is \( (W^T A V, W^T B, C V, D) \), where 
   \[
   W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \quad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.
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Note:
\[
V^T W = (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}})
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**Balanced Truncation**

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$$= \Sigma_1^{-\frac{1}{2}} [I_r, 0] \begin{bmatrix} \Sigma_1 & \_ \_ \\ \_ \_ & \Sigma_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \Sigma_1^{-\frac{1}{2}}$$
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$\implies VW^T$ is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.
Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$. 
Balanced Truncation

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of $r$ via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum_{k=r+1}^{n} \sigma_k \right) \|u\|_2.$$
Properties:

General misconception: complexity $O(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).
Balanced Truncation

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Use low-rank techniques ideas from numerical linear algebra:
Balanced Truncation

Properties:

**General misconception:** complexity $O(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

Use low-rank techniques ideas from numerical linear algebra:

- Instead of Gramians $P, Q$ compute $S, R \in \mathbb{R}^{n \times k}, k \ll n$, such that

  $$P \approx SS^T, \quad Q \approx RR^T.$$ 

- Compute $S, R$ with problem-specific Lyapunov solvers of “low” complexity directly.
Properties:

General misconception: complexity $O(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

Use low-rank techniques ideas from numerical linear algebra:

**Sparse Balanced Truncation:**

- Implementation using sparse Lyapunov solver ($\rightarrow$ ADI+sparse LU).
- Complexity $O(n(k^2 + r^2))$.
- Software:
  - MATLAB toolbox LyaPack (Penzl 1999),
  - Software library M.E.S.S.\(^a\) in C/MATLAB [B./SaaK/Köhler/uvm.],
  - pyMOR.

\(^a\)Matrix Equation Sparse Solvers
Recall Peaceman-Rachford ADI:

Consider $Au = s$ where $A \in \mathbb{R}^{n \times n}$ spd, $s \in \mathbb{R}^n$.

**ADI iteration idea:** decompose $A = H + V$ with $H, V \in \mathbb{R}^{n \times n}$ such that

$$
(H + pl)v = r \\
(V + pl)w = t
$$

can be solved easily/efficiently.
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**ADI Iteration**

If \( H, V \) spd \( \Rightarrow \exists p_k, k = 1, 2, \ldots, \) such that

\[
\begin{align*}
u_0 &= 0 \\
(H + p_k l)u_{k-\frac{1}{2}} &= (p_k l - V)u_{k-1} + s \\
(V + p_k l)u_k &= (p_k l - H)u_{k-\frac{1}{2}} + s
\end{align*}
\]

converges to \( u \in \mathbb{R}^n \) solving \( Au = s \).
The Lyapunov operator

\[ \mathcal{L} : \ P \ \mapsto \ AX + XA^T \]

can be decomposed into the linear operators

\[ \mathcal{L}_H : X \mapsto AX, \quad \mathcal{L}_V : X \mapsto XA^T. \]

In analogy to the standard ADI method we find the

**ADI iteration for the Lyapunov equation**  

[Wachspress 1988]

\[
\begin{align*}
X_0 &= 0, \\
(A + p_k I)X_{k-\frac{1}{2}} &= -W - X_{k-1}(A^T - p_k I), \\
(A + p_k I)X_k^T &= -W - X_{k-\frac{1}{2}}^T(A^T - p_k I).
\end{align*}
\]
Consider $AX + XA^T = -BB^T$ for stable $A$, $B \in \mathbb{R}^{n \times m}$ with $m \ll n$.

**ADI iteration for the Lyapunov equation**

For $k = 1, \ldots, k_{\text{max}}$

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\end{align*}
\]

Rewrite as one step iteration and factorize $X_k = Z_kZ_k^T$, $k = 0, \ldots, k_{\text{max}}$

\[
\begin{align*}
Z_0Z_0^T &= 0 \\
Z_kZ_k^T &= -2p_k(A + p_k I)^{-1}BB^T(A + p_k I)^{-T} \\
&\quad + (A + p_k I)^{-1}(A - p_k I)Z_{k-1}Z_{k-1}^T(A - p_k I)^T(A + p_k I)^{-T}
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Consider $AX + XA^T = -BB^T$ for stable $A$, $B \in \mathbb{R}^{n \times m}$ with $m \ll n$.

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\end{align*}
\]

$\ldots \leadsto$ low-rank Cholesky factor ADI

$Z_k = \left[ \sqrt{-2p_k} (A + p_k I)^{-1} B, \ (A + p_k I)^{-1} (A - p_k I) Z_{k-1} \right]$ [PENZL '00]
\[ Z_k = \left[ \sqrt{-2p_k}(A + p_k I)^{-1}B, \ (A + p_k I)^{-1}(A - p_k I)Z_{k-1} \right] \quad \text{[Penzl '00]} \]

Observing that \((A - p_i I), (A + p_k I)^{-1}\) commute, we rewrite \(Z_{k_{\max}}\) as

\[ Z_{k_{\max}} = [z_{k_{\max}}, P_{k_{\max}-1}z_{k_{\max}}, P_{k_{\max}-2}(P_{k_{\max}-1}z_{k_{\max}}), \ldots, P_1(P_2 \cdots P_{k_{\max}-1}z_{k_{\max}})] \]

where

\[ z_{k_{\max}} = \sqrt{-2p_{k_{\max}}}(A + p_{k_{\max}} I)^{-1}B \]

and

\[ P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}} \left[ I - (p_i + p_{i+1})(A + p_i I)^{-1} \right]. \quad \text{[Li/White '02]} \]
$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B, (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$  \hfill [PENZL '00]

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$z_{k_{\text{max}}} = \sqrt{-2p_{k_{\text{max}}}(A + p_{k_{\text{max}}} I)^{-1}B}$

and

$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}} [I - (p_i + p_{i+1})(A + p_i I)^{-1}]$ .  \hfill [LI/WHITE '02]

Need to solve only one (sparse) linear system with $m$ right-hand sides per iteration!

\[ V_1 \leftarrow \sqrt{-2 \text{Re} p_1 (A + p_1 I)^{-1} B}, \quad Z_1 \leftarrow V_1 \]

**FOR** \( k = 2, 3, \ldots \)

\[ V_k \leftarrow \sqrt{\frac{\text{Re} p_k}{\text{Re} p_{k-1}}} \left( V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1} \right) \]

\[ Z_k \leftarrow \begin{bmatrix} Z_{k-1} & V_k \end{bmatrix} \]

\[ Z_k \leftarrow \text{rrlq}(Z_k, \tau) \quad \% \text{column compression, optional} \]

At convergence, \( Z_k \max Z_k^T \approx X \), where (without column compression)

\[ Z_k \max = [V_1 \ldots V_k \max] \quad V_k = \in \mathbb{C}^{n \times m} \].

Note: Implementation in real arithmetic is possible: combine two steps [B./Li/Penzl 1999/2008] or employ the relations of consecutive complex factors [B./Kürschner/Saak 2011].

Current implementations (pyMOR, M.E.S.S.) employ low-rank property of residual, update residual in each step, and compute new shifts on the fly!

\[ V_1 \leftarrow \sqrt{-2 \text{re} \, p_1 (A + p_1 I)^{-1} B}, \quad Z_1 \leftarrow V_1 \]

FOR \( k = 2, 3, \ldots \)

\[ V_k \leftarrow \sqrt{\frac{\text{re} \, p_k}{\text{re} \, p_{k-1}}} \left( V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1} \right) \]

\[ Z_k \leftarrow \begin{bmatrix} Z_{k-1} & V_k \end{bmatrix} \]

\[ Z_k \leftarrow \text{rrlq}(Z_k, \tau) \quad \% \text{column compression, optional} \]

At convergence, \( Z_{k_{\text{max}}} Z_{k_{\text{max}}}^T \approx X \), where (without column compression)

\[ Z_{k_{\text{max}}} = \begin{bmatrix} V_1 & \ldots & V_{k_{\text{max}}} \end{bmatrix}, \quad V_k = \begin{bmatrix} \end{bmatrix} \in \mathbb{C}^{n \times m}. \]

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**Note:** Implementation in real arithmetic is possible: combine two steps [B./Li/Penzl 1999/2008] or employ the relations of consecutive complex factors [B./Kürschner/Saak 2011].

**Current implementations (pyMOR, M.E.S.S.)** employ low-rank property of residual, update residual in each step, and compute new shifts on the fly!
Numerical Results for ADI
Optimal Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

\[ c \cdot \rho \frac{\partial x}{\partial t} = \lambda \Delta x, \quad \xi \in \Omega \]
\[ \lambda \frac{\partial x}{\partial n} = \kappa (u_k - x), \quad \xi \in \Gamma_k, \ 1 \leq k \leq 7, \]
\[ \frac{\partial x}{\partial n} = 0, \quad \xi \in \Gamma_7. \]

\[ \Rightarrow \ m = 7, \ p = 6. \]

- FEM Discretization, different models for initial mesh (n = 371),
1, 2, 3, 4 steps of mesh refinement \( \Rightarrow \)
\[ n = 1357, 5177, 20209, 79841. \]

Source: Physical model: courtesy of Mannesmann/Demag.
Solve dual Lyapunov equations needed for balanced truncation, i.e.,

\[ AP\lambda M^T + M\lambda PA^T + BB^T = 0, \quad A^T QM + M^T QA + C^TC = 0, \]

for 79, 841.

25 shifts chosen by Penzl heuristic from 50/25 Ritz values of A of largest/smallest magnitude, no column compression performed.

M.E.S.S. requires no factorization of mass matrix.

Computations done on Core2Duo at 2.8GHz with 3GB RAM and 32Bit-MATLAB.
Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\text{range}(Z), Z \in \mathbb{R}^{n \times r}$, for subspace $Z \subset \mathbb{R}^n$, $\dim Z = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

\[
Z = \mathcal{K}(A, B, r) = \text{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}
\]

Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis range($Z$), $Z \in \mathbb{R}^{n \times r}$, for subspace $Z \subset \mathbb{R}^n$, $\dim Z = r$.
2. Set $\hat{A} := Z^T A Z$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A} \hat{X} + \hat{X} \hat{A}^T + \hat{B} \hat{B}^T = 0$.
4. Use $X \approx Z \hat{X} Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$Z = \mathcal{K}(A, B, r) = \text{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}$$


- Extended (and rational) Krylov method (EKSM, RKSM) [Simoncini 2007, Druskin/Knizhnerman/Simoncini 2011],

$$Z = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).$$
Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\text{range}(Z), Z \in \mathbb{R}^{n \times r}$, for subspace $Z \subset \mathbb{R}^n$, $\dim Z = r$.
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4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- ADI subspace [B./R.-C. Li/Truhar 2008]:

  $$Z = \text{colspan} \left[ V_1, \ldots, V_r \right].$$

Note:

1. ADI subspace is rational Krylov subspace [J.-R. Li/White 2002].
$n = 1357$, Absolute Error

- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.
Balanced Truncation

Numerical example for BT: Optimal Cooling of Steel Profiles

$n = 1357$, **Absolute Error**

- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

$n = 79841$, **Absolute Error**

- BT model computed using M.E.S.S. in MATLAB,
- dualcore, computation time: <10 min.
Balanced Truncation
Numerical example for BT: Microgyroscope (Butterfly Gyro)

- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

Vibrating micro-mechanical gyroscope for inertial navigation.
Rotational position sensor.

Source: http://modelreduction.org/index.php/Modified_Gyroscope
FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
\[ n = 34,722, \ m = 1, \ p = 12. \]

Reduced model computed using \texttt{SPARED}, \( r = 30. \)
Balanced Truncation
Numerical example for BT: Microgyroscope (Butterfly Gyro)

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  \[ n = 34,722, \quad m = 1, \quad p = 12. \]
- Reduced model computed using \texttt{SpaRed}, \( r = 30 \).

**Frequency Response Analysis**

![Bode Diagram](image)
FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
\[ n = 34,722, \quad m = 1, \quad p = 12. \]

Reduced model computed using \textit{SpaRed}, \( r = 30. \)
Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$. 
Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$.

Classical Balanced Truncation (BT) [Mullis/Roberts 1976, Moore 1981]

- $P =$ controllability Gramian of system given by $(A, B, C, D)$.
- $Q =$ observability Gramian of system given by $(A, B, C, D)$.
- $P$, $Q$ solve dual Lyapunov equations

\[
AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.
\]
Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$.

LQG Balanced Truncation (LQGBT) \[\text{[Jonckheere/Silverman 1983]}\]

- $P/Q = \text{controllability/observability Gramian of closed-loop system based on LQG compensator.}$
- $P, Q$ solve dual algebraic Riccati equations (AREs)

\[
0 = AP + PA^T - PC^T CP + B^T B, \\
0 = A^T Q + QA - QBB^T Q + C^T C.
\]
Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$.

Balanced Stochastic Truncation (BST)  

- $P =$ controllability Gramian of system given by $(A, B, C, D)$, i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by $(A, B, C, D)$, i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1} C$, $B_W := BD^T + PC^T$. 
Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT)

Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.

- $P, Q$ solve dual AREs
  $$0 = \bar{A}P + P\bar{A}^T + PC^T \bar{R}^{-1} CP + B\bar{R}^{-1} B^T,$$
  $$0 = \bar{A}^T Q + Q\bar{A} + QB\bar{R}^{-1} B^T Q + C^T \bar{R}^{-1} C,$$
  where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1} C$. 

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### Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$ P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0, $$

and truncate corresponding realization at size $r$ with $\sigma_r > \sigma_{r+1}$.

### Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere ’88];
- $H_\infty$ balanced truncation (HinfBT) – closed-loop balancing based on $H_\infty$ compensator [Mustafa/Glover ’91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.
Guaranteed preservation of physical properties like

- stability (all),
- passivity (PRBT),
- minimum phase (BST).

Computable error bounds, e.g.,

\[ \| G - G_r \|_\infty \leq 2n \sum_{j=r+1}^{\sigma_{BT}} \sigma_{BT} j, \]

\[ \| G - G_r \|_\infty \leq 2n \sum_{j=r+1}^{\sigma_{LQG}} \sigma_{LQG} j \sqrt{1 + (\sigma_{LQG} j)^2}, \]

\[ \| G - G_r \|_\infty \leq (n \prod_{j=r+1}^{\sigma_{BST}} 1 + \sigma_{BST} j 1 - \sigma_{BST} j - 1) \| G \|_\infty, \]

Can be combined with singular perturbation approximation (= Guyan reduction applied to balanced realization!) for improved steady-state performance.

Computations can be modularized \( \Rightarrow \) software packages M-M.E.S.S., MORLAB, see http://www.mpi-magdeburg.mpg.de/823508/software.
Balancing-Related Model Reduction

Properties

- Guaranteed preservation of physical properties like
  - stability (all),

Computable error bounds, e.g.,
- BT:
  \[ \| G - G_r \|_\infty \leq 2n \sum_{j=r+1}^\infty \sigma_{BT,j} \],
- LQGBT:
  \[ \| G - G_r \|_\infty \leq 2n \sum_{j=r+1}^\infty \sigma_{LQG,j} \sqrt{1 + (\sigma_{LQG,j})^2} \],
- BST:
  \[ \| G - G_r \|_\infty \leq \left( n \prod_{j=r+1}^\infty 1 + \sigma_{BST,j} \right)^{1/2} \| G \|_\infty \],

Can be combined with singular perturbation approximation (\(=\) Guyan reduction applied to balanced realization!) for improved steady-state performance.

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\]
\[
\|G - G_r\|_\infty \leq 2n \sum_{j=r+1}^{\infty} \sigma_{LQG_j} \sqrt{1 + (\sigma_{LQG_j})^2},
\]
\[
\|G - G_r\|_\infty \leq \left( \prod_{j=r+1}^{\infty} 1 + \sigma_{BST_j} - 1 \right) \|G\|_\infty,
\]

Can be combined with singular perturbation approximation ( = Guyan reduction applied to balanced realization!) for improved steady-state performance.

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Computable error bounds, e.g.,
- \( \| G - G_r \|_{\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_{BT,j} \),
- \( \| G - G_r \|_{\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_{LQG,j} \sqrt{1 + (\sigma_{LQG,j})^2} \),
- \( \| G - G_r \|_{\infty} \leq \left( \prod_{j=r+1}^{n} 1 + \sigma_{BST,j} \right) \| G \|_{\infty} \).

Can be combined with singular perturbation approximation (＝ Guyan reduction applied to balanced realization!) for improved steady-state performance.

Computations can be modularized \( \Rightarrow \) software packages M-M.E.S.S., MORLAB, see [link].
Balancing-Related Model Reduction

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  - minimum phase (BST).

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\[ \| G - G_r \|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT}, \]

\[ \| G - G_r \|_\infty \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}} \]

\[ \| G - G_r \|_\infty \leq \left( \prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \| G \|_\infty, \]

Can be combined with singular perturbation approximation ( = Guyan reduction applied to balanced realization!) for improved steady-state performance.

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$$\text{BST: } \|G - G_r\|_{\infty} \leq \left( \prod_{j=r+1}^{n} \frac{1 + \sigma_j^{BST}}{1 - \sigma_j^{BST}} - 1 \right) \|G\|_{\infty} ,$$

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\[
\begin{align*}
\text{BT: } \| G - G_r \|_\infty & \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT}, \\
\text{LQGBT: } \| G - G_r \|_\infty & \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}} \\
\text{BST: } \| G - G_r \|_\infty & \leq (\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1) \| G \|_\infty,
\end{align*}
\]

- Can be combined with singular perturbation approximation ( = Guyan reduction applied to balanced realization!) for improved steady-state performance.
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1. Introduction

2. Model Reduction by Projection

3. Balanced Truncation

4. Final Remarks
Special methods for second-order (mechanical) and delay systems.

Extensions to bilinear, quadratic-bilinear, polynomial, and stochastic systems.

Empirical variants using snapshots \(\sim\) ICERM semester visitor Christian Himpe!

MOR methods for discrete-time systems.

Extensions to descriptor systems \(E \dot{x} = Ax + Bu, E\) singular.

Parametric model reduction:

\[
\dot{x} = A(p)x + B(p)u, \quad y = C(p)x,
\]

where \(p \in \mathbb{R}^d\) is a free parameter vector; parameters should be preserved in the reduced-order model.
References

  *Model Reduction for Control System Design.*  

- P. Benner, E.S. Quintana-Ortí, and G. Quintana-Ortí.  
  State-space truncation methods for parallel model reduction of large-scale systems.  

- P. Benner, V. Mehrmann, and D. Sorensen (editors).  
  *Dimension Reduction of Large-Scale Systems.*  

- A.C. Antoulas.  
  *Approximation of Large-Scale Dynamical Systems.*  

- P. Benner.  
  Numerical linear algebra for model reduction in control and simulation.  

- W.H.A. Schilders, H.A. van der Vorst, and J. Rommes (editors).  
  *Model Order Reduction: Theory, Research Aspects and Applications.*  

- P. Benner, J. ter Maten, and M. Hinze (editors).  
  *Model Reduction for Circuit Simulation.*  

  Model order reduction for linear and nonlinear systems: a system-theoretic perspective.  

  *Model Reduction and Approximation: Theory and Algorithms.*  