MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

AN INTRODUCTION TO SYSTEM-THEORETIC METHODS FOR MODEL REDUCTION

Part I: Balancing-based Methods

Peter Benner

January 30, 2020

Special Semester on
“Model and dimension reduction in uncertain and dynamic systems”
ICERM at Brown University

Outline

Outline

Problem

Given a physical problem with dynamics described by the states $x \in \mathbb{R}^n$, *where* n *is the dimension of the state space.*

Problem

Given a physical problem with dynamics described by the states $x \in \mathbb{R}^n$, where n is the dimension of the state space.

Because of redundancies, complexity, etc., we want to describe the dynamics of the system using a reduced number of states.

Problem

Given a physical problem with dynamics described by the states $x \in \mathbb{R}^n$, where n is the dimension of the state space.

Because of redundancies, complexity, etc., we want to describe the dynamics of the system using a reduced number of states.

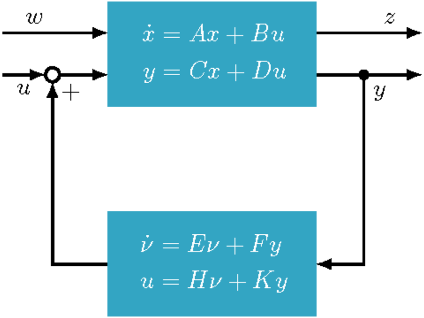
This is the task of model reduction (also: dimension reduction, order reduction).

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



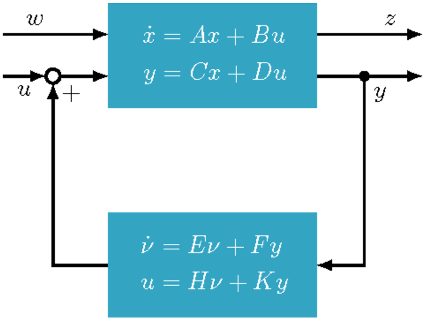
Application Areas
(Optimal) Control

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



Practical controllers require small N ($N \sim 10$, say) due to

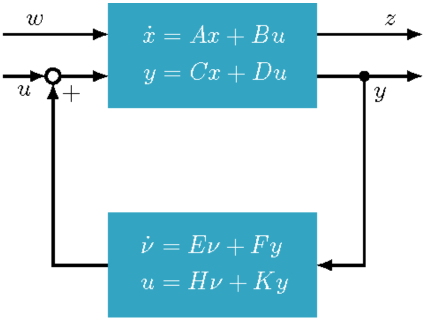
- real-time constraints,
- increasing fragility for larger N.

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N.

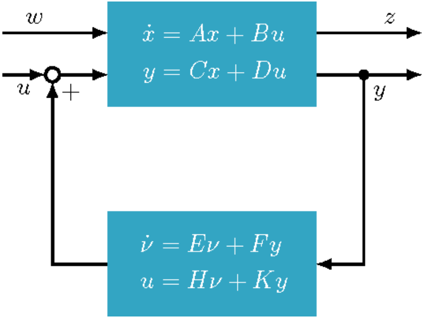
$\Longrightarrow$ reduce order of plant (n) and/or controller (N).

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
- output $=$ input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N.

$\Longrightarrow$ reduce order of plant (n) and/or controller (N).

Standard MOR techniques in systems and control: balanced truncation and related methods.

Application Areas

Micro Electronics/Circuit Simulation

- Progressive miniaturization: **Moore's Law** states that the number of on-chip transistors doubles each 12 (now: 18) months.

- Progressive miniaturization: **Moore's Law** states that the number of on-chip transistors doubles each 12 (now: 18) months.
- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.

- Progressive miniaturization: **Moore's Law** states that the number of on-chip transistors doubles each 12 (now: 18) months.
- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- Increase in packing density requires modeling of interconncet to ensure that thermic/electro-magnetic effects do not disturb signal transmission.

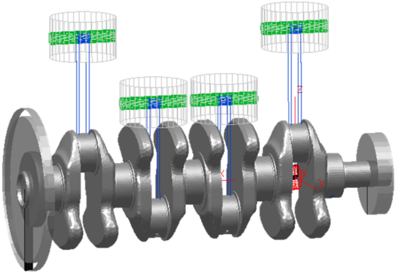
Application Areas

Micro Electronics/Circuit Simulation

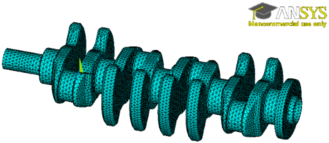
- Progressive miniaturization: **Moore's Law** states that the number of on-chip transistors doubles each 12 (now: 18) months.
- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- Increase in packing density requires modeling of interconncet to ensure that thermic/electro-magnetic effects do not disturb signal transmission.
- Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines (interconnect, powergrid), parasitic effects,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

- Progressive miniaturization: **Moore's Law** states that the number of on-chip transistors doubles each 12 (now: 18) months.
- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- Increase in packing density requires modeling of interconncet to ensure that thermic/electro-magnetic effects do not disturb signal transmission.
- Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines (interconnect, powergrid), parasitic effects,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

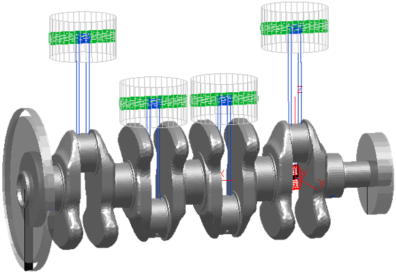
Standard MOR techniques in circuit simulation: Krylov subspace / Padé approximation / rational interpolation methods.



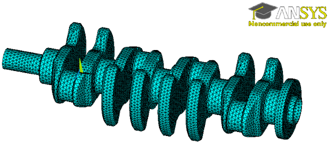
$\rightsquigarrow$



- Resolving complex 3D geometries $\Rightarrow$ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.



$\rightsquigarrow$



- Resolving complex 3D geometries $\Rightarrow$ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.

Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) $\rightsquigarrow$ Craig-Bampton method — not discussed in this course!

An Inspiration: Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory: $4 \cdot n_x \cdot n_y$ bytes.

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory: $4 \cdot n_x \cdot n_y$ bytes.

Theorem: (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U \Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\left\| X - \widehat{X} \right\|_2 = \sigma_{r+1}$.

An Inspiration: Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory: $4 \cdot n_x \cdot n_y$ bytes.

Theorem: (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

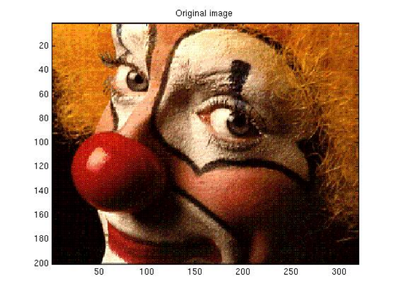
where $X = U\Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\left\|X - \widehat{X}\right\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \dots, u_r$, $\sigma_1 v_1, \dots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

Example: Image Compression by Truncated SVD

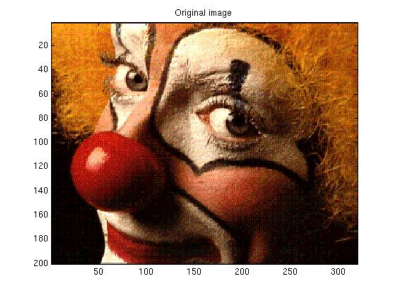
Example: Clown



320×200 pixel

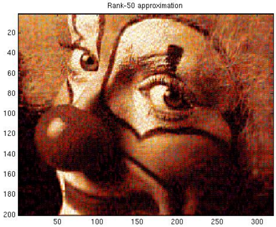
$\rightsquigarrow \quad \approx 256$ kb

Example: Clown

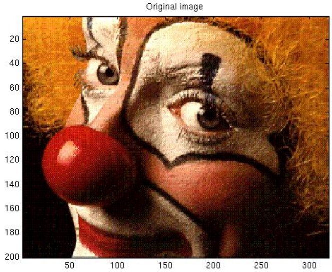


320×200 pixel
$\rightsquigarrow \quad \approx 256$ kb

- rank $r = 50$, ≈ 104 kb



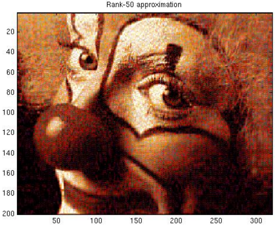
Example: Clown



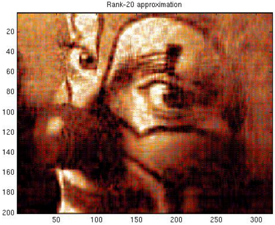
320×200 pixel

$\rightsquigarrow \quad \approx 256$ kb

- rank $r = 50$, ≈ 104 kb

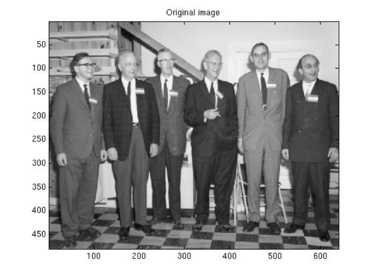


- rank $r = 20$, ≈ 42 kb



Example: Gatlinburg

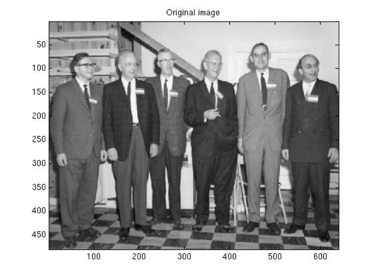
Organizing committee
Gatlinburg/Householder Meeting 1964:
James H. Wilkinson, Wallace Givens, George Forsythe, Alston Householder, Peter Henrici, Fritz L. Bauer.



640×480 pixel, ≈ 1229 kb

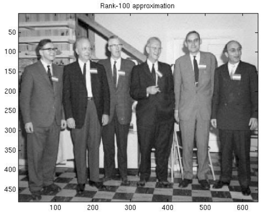
Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:
James H. Wilkinson, Wallace Givens, George Forsythe, Alston Householder, Peter Henrici, Fritz L. Bauer.

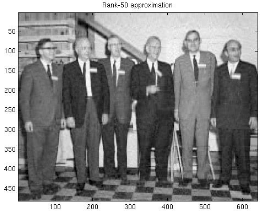


640×480 pixel, ≈ 1229 kb

- rank $r = 100$, ≈ 448 kb



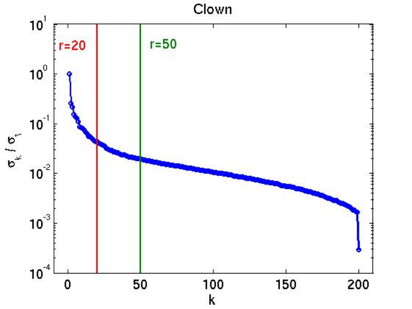
- rank $r = 50$, ≈ 224 kb

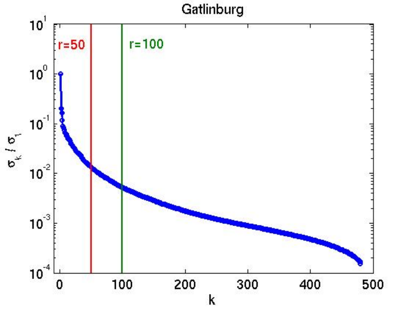


Background: Singular Value Decay

Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices





Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) = g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

Original System

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \, \|u\|$ for all admissible input signals.

Model Reduction for Dynamical Systems

Original System

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

Reduced-Order System

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^p$.

Goal:

$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.

Model Reduction for Dynamical Systems

Original System

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

Reduced-Order System

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^p$.

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Original System

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t), u(t)). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

Reduced-Order System

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^p$.

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= f(t,x,u) &= Ax + Bu, &\qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= g(t,x,u) &= Cx + Du, &\qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

Linear, Time-Invariant (LTI) Systems

$$
\begin{aligned}
\dot{x} &= f(t,x,u) = Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\
y &= g(t,x,u) = Cx + Du, \qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}.
\end{aligned}
$$

Assumptions (for now): $t_0 = 0$, $x_0 = x(0) = 0$, $D = 0$.

Linear, Time-Invariant (LTI) Systems

$$\begin{array}{rclcll} \dot{x} & = & f(t,x,u) & = & Ax + Bu, & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y & = & g(t,x,u) & = & Cx + Du, & C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{array}$$

State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\, d\tau \quad \text{for all } t \in \mathbb{R}.$$

Linear, Time-Invariant (LTI) Systems

$$\begin{array}{rclcl} \dot{x} & = & f(t,x,u) & = & Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y & = & g(t,x,u) & = & Cx + Du, \qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{array}$$

State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t \in \mathbb{R}.$$

- $\mathcal{S} : \mathcal{U} \to \mathcal{Y}$ is a linear operator between (function) spaces.

Linear, Time-Invariant (LTI) Systems

$$\begin{array}{rclcll} \dot{x} &=& f(t,x,u) &=& Ax + Bu, & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &=& g(t,x,u) &=& Cx + Du, & C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{array}$$

State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t \in \mathbb{R}.$$

- $\mathcal{S} : \mathcal{U} \to \mathcal{Y}$ is a linear operator between (function) spaces.
- Recall: matrix in $\mathbb{R}^{n\times m}$ is a linear operator, mapping $\mathbb{R}^m \to \mathbb{R}^n$!

Linear, Time-Invariant (LTI) Systems

$$
\begin{array}{rclcll}
\dot{x} & = & f(t,x,u) & = & Ax + Bu, & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\
y & = & g(t,x,u) & = & Cx + Du, & C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}.
\end{array}
$$

State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$
\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} C e^{A(t-\tau)} B u(\tau)\, d\tau \quad \text{for all } t \in \mathbb{R}.
$$

- $\mathcal{S} : \mathcal{U} \to \mathcal{Y}$ is a linear operator between (function) spaces.
- Recall: matrix in $\mathbb{R}^{n\times m}$ is a linear operator, mapping $\mathbb{R}^m \to \mathbb{R}^n$!
- Basic Idea: use SVD approximation as for matrix A!

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= f(t,x,u) &= Ax + Bu, \qquad & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= g(t,x,u) &= Cx + Du, \qquad & C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\, d\tau \quad \text{for all } t \in \mathbb{R}.$$

- $\mathcal{S} : \mathcal{U} \to \mathcal{Y}$ is a linear operator between (function) spaces.
- Recall: matrix in $\mathbb{R}^{n\times m}$ is a linear operator, mapping $\mathbb{R}^m \to \mathbb{R}^n$!
- Basic Idea: use SVD approximation as for matrix A!
- Problem: in general, $\mathcal{S}$ does not have a discrete SVD and can therefore not be approximated as in the matrix case!

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx, \qquad & C \in \mathbb{R}^{p\times n}. \end{aligned}$$

Alternative to State-Space Operator: Hankel operator

Instead of

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} C e^{A(t-\tau)} B u(\tau)\, d\tau \quad \text{for all } t \in \mathbb{R}.$$

use Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} C e^{A(t-\tau)} B u(\tau)\, d\tau \quad \text{for all } t > 0.$$

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad && A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx, && C \in \mathbb{R}^{p\times n}. \end{aligned}$$

Alternative to State-Space Operator: Hankel operator

Instead of

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t \in \mathbb{R}.$$

use Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

$\mathcal{H}$ compact, finite-dimensional $\Rightarrow$ $\mathcal{H}$ has discrete SVD

$\rightsquigarrow$ *Hankel singular values* $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \ldots \geq \sigma_n \geq \sigma_{n+1} = 0 = \ldots = 0.$

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx, & C \in \mathbb{R}^{p\times n}. \end{aligned}$$

Alternative to State-Space Operator: Hankel operator

Instead of

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t \in \mathbb{R}.$$

use Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

$\mathcal{H}$ compact, finite-dimensional $\Rightarrow$ $\mathcal{H}$ has discrete SVD

$\rightsquigarrow$ *Hankel singular values* $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \ldots \geq \sigma_n \geq \sigma_{n+1} = 0 = \ldots = 0.$

$\Longrightarrow$ SVD-type approximation of $\mathcal{H}$ possible!

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx, & C \in \mathbb{R}^{p\times n}. \end{aligned}$$

Alternative to State-Space Operator: Hankel operator

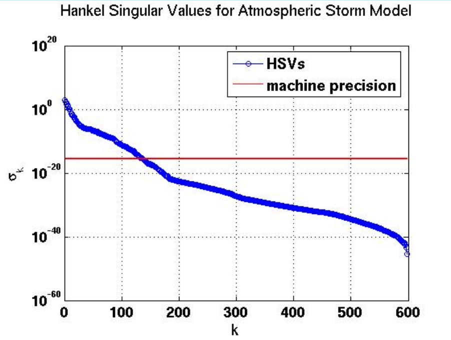
$\mathcal{H}$ compact

$\Downarrow$

$\mathcal{H}$ has discrete SVD

$\Downarrow$

Hankel singular values



Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad && A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx, && C \in \mathbb{R}^{p\times n}. \end{aligned}$$

Alternative to State-Space Operator: Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

$\mathcal{H}$ compact $\Rightarrow$ $\mathcal{H}$ has discrete SVD

$\Rightarrow$ Best approximation problem w.r.t. 2-induced operator norm well-posed

Linear, Time-Invariant (LTI) Systems

$$\begin{array}{rcll} \dot{x} & = & Ax + Bu, & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y & = & Cx, & C \in \mathbb{R}^{p\times n}. \end{array}$$

Alternative to State-Space Operator: Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

$\mathcal{H}$ compact $\Rightarrow$ $\mathcal{H}$ has discrete SVD

$\Rightarrow$ Best approximation problem w.r.t. 2-induced operator norm well-posed

$\Rightarrow$ solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

Linear, Time-Invariant (LTI) Systems

$$\begin{array}{rcll} \dot{x} & = & Ax + Bu, & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y & = & Cx, & C \in \mathbb{R}^{p\times n}. \end{array}$$

Alternative to State-Space Operator: Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

$\mathcal{H}$ compact $\Rightarrow$ $\mathcal{H}$ has discrete SVD

$\Rightarrow$ Best approximation problem w.r.t. 2-induced operator norm well-posed

$\Rightarrow$ solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

But: computationally infeasible for large-scale systems.

Linear Systems in Frequency Domain

Linear, Time-Invariant (LTI) Systems

$$\Sigma : \left\{ \begin{array}{lcl} \dot{x} & = & Ax + Bu, \\ y & = & Cx + Du, \end{array} \right. \qquad \begin{array}{ll} A \in \mathbb{R}^{n \times n}, & B \in \mathbb{R}^{n \times m}, \\ C \in \mathbb{R}^{p \times n}, & D \in \mathbb{R}^{p \times m}. \end{array}$$

Assumptions: $t_0 = 0$, $x_0 = x(0) = 0$.

Laplace Transform / Frequency Domain

Application of Laplace transform

$$\mathcal{L} : x(t) \mapsto x(s) = \int_0^\infty e^{-st} x(t)\, dt \quad (\Rightarrow \ \dot{x}(t) \mapsto sx(s))$$

with $s \in \mathbb{C}$ leads to linear system of equations:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s).$$

Linear Systems in Frequency Domain

Linear, Time-Invariant (LTI) Systems

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y = Cx + Du, & C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{cases}$$

Assumptions: $t_0 = 0$, $x_0 = x(0) = 0$.

Laplace Transform / Frequency Domain

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s)$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)} \Big) u(s) = G(s)u(s).$$

G is the transfer function of Σ, $G : \mathcal{L}_2^m \to \mathcal{L}_2^p \quad (\mathcal{L}_2 := \mathcal{L}(L_2(-\infty,\infty)))$.

Model Reduction as Approximation Problem

Approximation Problem

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}. \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \qquad \hat{A} \in \mathbb{R}^{r \times r}, \quad \hat{B} \in \mathbb{R}^{r \times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \qquad \hat{C} \in \mathbb{R}^{p \times r}, \quad \hat{D} \in \mathbb{R}^{p \times m}. \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \left\| Gu - \hat{G}u \right\| \leq \left\| G - \hat{G} \right\| \|u\| \leq \text{tolerance} \cdot \|u\| \,.$$

Approximation Problem

Approximate the dynamical system

$$\begin{array}{rcll} \dot{x} & = & Ax + Bu, & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y & = & Cx + Du, & C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{array}$$

by reduced-order system

$$\begin{array}{rcll} \dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r\times r}, \quad \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{p\times r}, \quad \hat{D} \in \mathbb{R}^{p\times m}. \end{array}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \left\|Gu - \hat{G}u\right\| \leq \left\|G - \hat{G}\right\| \|u\| \leq \text{tolerance} \cdot \|u\| \,.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G})\leq r} \left\|G - \hat{G}\right\|.$

Definition

A linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \,|\, \Re(z) < 0\}$.

Definition

A linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \,|\, \Re(z) < 0\}$.

Lemma

Sufficient for asymptotic stability is that A is asymptotically stable (or Hurwitz), i.e., the spectrum of A, denoted by $\Lambda(A)$, satisfies $\Lambda(A) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

Questions:

- For fixed $x_0 \in \mathbb{R}^n$ and some $x^1 \in \mathbb{R}^n$, is there a feasible control function $u \in \mathcal{U}_{ad}$ (e.g., $\mathcal{U}_{ad} \in \{C^k[0,T], L_2(0,T), PC[0,T]\}$, possibly with constraints $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$) and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$?
  What is the set of targets x^1 reachable from x^0?

Questions:

- For fixed $x_0 \in \mathbb{R}^n$ and some $x^1 \in \mathbb{R}^n$, is there a feasible control function $u \in \mathcal{U}_{ad}$ (e.g., $\mathcal{U}_{ad} \in \{C^k[0,T], L_2(0,T), PC[0,T]\}$, possibly with constraints $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$) and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$?
  What is the set of targets x^1 reachable from x^0?
- For fixed $x_1 \in \mathbb{R}^n$ and some $x^0 \in \mathbb{R}^n$, is there a feasible control function $u \in \mathcal{U}_{ad}$ and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$?
  What is the set of initial conditions x^0 controllable to x^1?

Questions:

- For fixed $x_0 \in \mathbb{R}^n$ and some $x^1 \in \mathbb{R}^n$, is there a feasible control function $u \in \mathcal{U}_{ad}$ (e.g., $\mathcal{U}_{ad} \in \{C^k[0,T], L_2(0,T), PC[0,T]\}$, possibly with constraints $\underline{u}(t) \le u(t) \le \overline{u}(t)$) and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$?
  What is the set of targets x^1 reachable from x^0?
- For fixed $x_1 \in \mathbb{R}^n$ and some $x^0 \in \mathbb{R}^n$, is there a feasible control function $u \in \mathcal{U}_{ad}$ and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$?
  What is the set of initial conditions x^0 controllable to x^1?

Note: for LTI systems $\dot{x} = Ax + Bu$, both concepts are equivalent!

Definition (Controllability)

Consider the target (the state to be reached) $x^1 \in \mathbb{R}^n$.

a) An LTI system with initial value $x(0) = x^0$ is controllable to x^1 in time $t_1 > 0$ if there exists $u \in \mathcal{U}_{ad}$ such that $x(t_1; u) = x^1$.

   (Equivalently, (t_1, x^1) is reachable from $(0, x^0)$.)

Definition (Controllability)

Consider the target (the state to be reached) $x^1 \in \mathbb{R}^n$.

a) An LTI system with initial value $x(0) = x^0$ is controllable to x^1 in time $t_1 > 0$ if there exists $u \in \mathcal{U}_{ad}$ such that $x(t_1; u) = x^1$.

   (Equivalently, (t_1, x^1) is reachable from $(0, x^0)$.)

b) x^0 is controllable to x^1 if there exists a $t_1 > 0$ such that (t_1, x^1) can be reached from $(0, x^0)$.

Definition (Controllability)

Consider the target (the state to be reached) $x^1 \in \mathbb{R}^n$.

a) An LTI system with initial value $x(0) = x^0$ is controllable to x^1 in time $t_1 > 0$ if there exists $u \in \mathcal{U}_{ad}$ such that $x(t_1; u) = x^1$.
   (Equivalently, (t_1, x^1) is reachable from $(0, x^0)$.)

b) x^0 is controllable to x^1 if there exists a $t_1 > 0$ such that (t_1, x^1) can be reached from $(0, x^0)$.

c) If the system is controllable to x^1 for all $x^0 \in \mathbb{R}^n$, it is (completely) controllable.

Definition (Controllability)

Consider the target (the state to be reached) $x^1 \in \mathbb{R}^n$.

a) An LTI system with initial value $x(0) = x^0$ is controllable to x^1 in time $t_1 > 0$ if there exists $u \in \mathcal{U}_{ad}$ such that $x(t_1; u) = x^1$.
   (Equivalently, (t_1, x^1) is reachable from $(0, x^0)$.)

b) x^0 is controllable to x^1 if there exists a $t_1 > 0$ such that (t_1, x^1) can be reached from $(0, x^0)$.

c) If the system is controllable to x^1 for all $x^0 \in \mathbb{R}^n$, it is (completely) controllable.

The controllability set w.r.t. x^1 is defined as $\mathcal{C} := \bigcup_{t_1 > 0} \mathcal{C}(t_1)$ where

$$\mathcal{C}(t_1) := \{x^0 \in \mathbb{R}^n \mid \exists u \in \mathcal{U}_{ad} : x(t_1; u) = x^1\}.$$

Definition (Controllability)

Consider the target (the state to be reached) $x^1 \in \mathbb{R}^n$.

a) An LTI system with initial value $x(0) = x^0$ is controllable to x^1 in time $t_1 > 0$ if there exists $u \in \mathcal{U}_{ad}$ such that $x(t_1; u) = x^1$.
   (Equivalently, (t_1, x^1) is reachable from $(0, x^0)$.)

b) x^0 is controllable to x^1 if there exists a $t_1 > 0$ such that (t_1, x^1) can be reached from $(0, x^0)$.

c) If the system is controllable to x^1 for all $x^0 \in \mathbb{R}^n$, it is (completely) controllable.

The controllability set w.r.t. x^1 is defined as $\mathcal{C} := \bigcup_{t_1 > 0} \mathcal{C}(t_1)$ where

$$\mathcal{C}(t_1) := \{x^0 \in \mathbb{R}^n \mid \exists u \in \mathcal{U}_{ad} : x(t_1; u) = x^1\}.$$

In short: an LTI system is controllable $\iff$ $\mathcal{C} = \mathbb{R}^n$.

Now: characterize controllability.

Now: characterize controllability.

Variation of constants $\Longrightarrow$

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}(x^0 + \int_0^t e^{-As}Bu(s)ds).$$

Now: characterize controllability.

Variation of constants $\Longrightarrow$

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}(x^0 + \int_0^t e^{-As}Bu(s)ds).$$

Hence, if x^0 is controllable to x^1:

$$x^1 = x(t_1) = e^{At_1}x^0 + \int_0^{t_1} e^{A(t_1-t)}Bu(t)dt$$

This is equivalent to

$$e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}Bu(t)dt.$$

Now: characterize controllability.

Variation of constants $\Longrightarrow$

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}(x^0 + \int_0^t e^{-As}Bu(s)ds).$$

Hence, if x^0 is controllable to x^1:

$$x^1 = x(t_1) = e^{At_1}x^0 + \int_0^{t_1} e^{A(t_1-t)}Bu(t)dt$$

This is equivalent to

$$e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}Bu(t)dt.$$

Ansatz: $u(t) = B^T e^{-A^T t}c \Longrightarrow$

$$e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}BB^T e^{-A^T t}dt c =: P(0, t_1)c.$$

Now: characterize controllability.

Variation of constants $\Longrightarrow$

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}(x^0 + \int_0^t e^{-As}Bu(s)ds).$$

Hence, if x^0 is controllable to x^1:

$$x^1 = x(t_1) = e^{At_1}x^0 + \int_0^{t_1} e^{A(t_1-t)}Bu(t)dt$$

This is equivalent to

$$e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}Bu(t)dt.$$

Ansatz: $u(t) = B^T e^{-A^T t}c \Longrightarrow$

$$e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}BB^T e^{-A^T t}dt c =: P(0, t_1)c.$$

Hence, an LTI system is controllable iff this linear system is solvable for $c \in \mathbb{R}^n$, i.e., iff $P(0, t_1)$ is invertible. (Note: $P(0, t_1) = P(0, t_1)^T \geq 0$ by definition!)

Now: characterize controllability.

Theorem

For an LTI system defined by $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, T.F.A.E.:

a) *The LTI system $\dot{x} = Ax + Bu$ is controllable.*

b) *The finite time Gramian $P(0, t_1)$ is spd $\forall\, t_1 > 0$.*

c) *The controllability matrix*

$$K(A, B) := [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times n\cdot m}$$

has full rank n. (Note: $\mathrm{range}(K(A, B)) = \mathcal{C}(t_1)\ \forall\, t_1 > 0$!)

d) *If z is a left eigenvector of A, then $z^*B \neq 0$.*

e) *(Hautus test)* $\mathrm{rank}([\lambda I - A, B]) = n\ \forall\, \lambda \in \mathbb{C}$.

The Gramian characterization of controllability for stable systems can be based on positive definiteness of the (infinite) controllability Gramian

$$P := \int_0^\infty e^{As} B B^T e^{A^T s} ds,$$

using congruence of $P(0, t_1)$ to $\int\limits_0^{t_1} e^{As} B B^T e^{A^T s} ds$ and taking the limit $t_1 \to \infty$.

The Gramian characterization of controllability for stable systems can be based on positive definiteness of the (infinite) controllability Gramian

$$P := \int_0^\infty e^{As} BB^T e^{A^T s} ds,$$

using congruence of $P(0, t_1)$ to $\int_0^{t_1} e^{As} BB^T e^{A^T s} ds$ and taking the limit $t_1 \to \infty$.

Theorem

For a stable LTI system defined by $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, T.F.A.E.:

a) *The LTI system $\dot{x} = Ax + Bu$ is controllable.*

b) *The controllability Gramian P is positive definite.*

New question: suppose we have

$$y(t) = \tilde{y}(t)$$

corresponding to two trajectories $x, \tilde{x}$ obtained by the same input function $u(t)$. Can we conclude that $x(0) = \tilde{x}(0)$, or even stronger, that $x(t) = \tilde{x}(t)$ for $t \leq 0, t \geq 0$ (past/future)?

(Note that $x(t_0) = \tilde{x}(t_0)$ is sufficient as trajectory uniquely determined. In other words, is the mapping $x^0 \rightarrow y(t)$ injective?)

New question: suppose we have

$$y(t) = \tilde{y}(t)$$

corresponding to two trajectories $x, \tilde{x}$ obtained by the same input function $u(t)$. Can we conclude that $x(0) = \tilde{x}(0)$, or even stronger, that $x(t) = \tilde{x}(t)$ for $t \le 0, t \ge 0$ (past/future)?

(Note that $x(t_0) = \tilde{x}(t_0)$ is sufficient as trajectory uniquely determined. In other words, is the mapping $x^0 \to y(t)$ injective?)

Definition (Observability)

An LTI system is reconstructable (observable) if for solution trajectories $x(t), \tilde{x}(t)$ obtained with the same input function u, we have

$$\begin{aligned} y(t) = &\ \tilde{y}(t) \quad \forall t \le 0 \quad (\forall t \ge 0) \\ \Longrightarrow \quad x(t) = &\ \tilde{x}(t) \quad \forall t \le 0 \quad (\forall t \ge 0). \end{aligned}$$

Characterization of observability/reconstructability:

Theorem (Duality)

An LTI system is reconstructable if and only if the dual system $\dot{x}(t) = -A^T x(t) - C^T u(t)$ *is controllable.*

Characterization of observability/reconstructability:

Theorem (Duality)

An LTI system is reconstructable if and only if the dual system $\dot{x}(t) = -A^T x(t) - C^T u(t)$ *is controllable.*

Theorem

For an LTI system defined by $(A, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times n}$, *T.F.A.E.:*

a) *The LTI system is reconstructable.*

b) *The LTI system is observable.*

c) *The observability matrix*

$$\mathcal{O}(A, C) = \left[C^T, A^T C^T, (A^2)^T C, \ldots, (A^{n-1})^T C^T\right]^T \in \mathbb{R}^{np\times n} \text{ has } \mathrm{rank}\ n.$$

d) *If* $Ax = \lambda x$, *then* $C^T x \neq 0$.

e) *(Hautus test)* $\mathrm{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$.

Characterization of observability/reconstructability:

Theorem (Duality)

An LTI system is reconstructable if and only if the dual system $\dot{x}(t) = -A^T x(t) - C^T u(t)$ *is controllable.*

Theorem

A stable LTI system is observable if and only if the observability Gramian

$$Q := \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

is symmetric positive definite.

- Controllability/observability are sometimes too strong.

- Controllability/observability are sometimes too strong.
- Weaker requirement: is there $u \in \mathcal{U}_{ad}$ to steer x_0 to vicinity of x^1?

- Controllability/observability are sometimes too strong.
- Weaker requirement: is there $u \in \mathcal{U}_{ad}$ to steer x_0 to vicinity of x^1?
- For LTI systems, it suffices to consider $x^1 = 0$!

- Controllability/observability are sometimes too strong.
- Weaker requirement: is there $u \in \mathcal{U}_{ad}$ to steer x_0 to vicinity of x^1?
- For LTI systems, it suffices to consider $x^1 = 0$!
- Hence, is there $u \in \mathcal{U}_{ad}$ so that $\lim_{t\to\infty} x(t;u) = 0$ ($\forall\, x^0 \in \mathbb{R}^n$)?

- Controllability/observability are sometimes too strong.
- Weaker requirement: is there $u \in \mathcal{U}_{ad}$ to steer x_0 to vicinity of x^1?
- For LTI systems, it suffices to consider $x^1 = 0$!
- Hence, is there $u \in \mathcal{U}_{ad}$ so that $\lim_{t\to\infty} x(t;u) = 0$ $(\forall\ x^0 \in \mathbb{R}^n)$?
- If the answer is **yes**, then the LTI system is called stabilizable

- Controllability/observability are sometimes too strong.
- Weaker requirement: is there $u \in \mathcal{U}_{ad}$ to steer x_0 to vicinity of x^1?
- For LTI systems, it suffices to consider $x^1 = 0$!
- Hence, is there $u \in \mathcal{U}_{ad}$ so that $\lim_{t\to\infty} x(t;u) = 0$ $(\forall\, x^0 \in \mathbb{R}^n)$?
- If the answer is **yes**, then the LTI system is called stabilizable

Theorem

For an LTI system defined by $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, T.F.A.E.:

a) *The LTI system is stabilizable.*

b) $\exists$ *feedback operator/matrix* $F \in \mathbb{R}^{m\times n}$ *with* $\Lambda(A + BF) \subset \mathbb{C}^-$.

c) *If* $p^*A = \tilde{\lambda}p^*$ *and* $\mathrm{Re}(\lambda) \geq 0$, *then* $p^*B \neq 0$.

d) $\mathrm{rank}([A - \lambda I, B]) = n \quad \forall \lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) \geq 0$.

e) $\Lambda(A_3) \subset \mathbb{C}^-$ *in the (controllability) Kalman decomposition of* (A, B),

$$V^T A V = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, V^T B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

$\exists$ dual concept of stabilizability, analogous to duality of controllability and observability.

Definition (Detectability)

An LTI system is detectable if for any solution $x(t)$ of $\dot{x} = Ax$ with $Cx(t) \equiv 0$ we have $\lim_{t\to\infty} x(t) = 0$.
(We can not observe all of x, but the unobservable part is stable.)

$\exists$ dual concept of stabilizability, analogous to duality of controllability and observability.

Theorem

For an LTI system defined by $(A, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times n}$, T.F.A.E.:

a) *The LTI system is detectable.*

b) (A^T, C^T) *is stabilizable.*

c) $Ax = \lambda x, \mathrm{Re}(\lambda) \geq 0 \Rightarrow C^T x \neq 0.$

d) $\mathrm{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ *for all* $\lambda, \mathrm{Re}(\lambda) \geq 0.$

e) *In the observability Kalman decomposition of* (A^T, C^T),

$$W^T A W = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, CW = [C_1 \ 0],$$

we have $\Lambda(A_3) \subset \mathbb{C}^-$.

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad \text{with transfer function} \\ y(t) &= Cx(t) + Du(t), \quad G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ is called a realization of Σ.

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with transfer function} \\ y(t) = Cx(t) + Du(t), & G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ is called a realization of Σ.

Realizations are not unique!

Transfer function is invariant under state-space transformations,

$$\mathcal{T} : \begin{cases} x \rightarrow Tx, \\ (A, B, C, D) \rightarrow (TAT^{-1}, TB, CT^{-1}, D), \end{cases}$$

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad \text{with transfer function} \\ y(t) &= Cx(t) + Du(t), \quad G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ is called a realization of Σ.

Realizations are not unique!

Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t),$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t),$$

for arbitrary $A_j \in \mathbb{R}^{n_j \times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $C_2 \in \mathbb{R}^{q \times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with transfer function} \\ y(t) = Cx(t) + Du(t), & G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ is called a realization of Σ.

Realizations are not unique!

Hence,

$$(A, B, C, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ!

Definition

For a linear (time-invariant) system

$$\Sigma: \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t), \\ y(t) & = & Cx(t) + Du(t), \end{array} \right. \quad \begin{array}{l} \text{with transfer function} \\ G(s) = C(sI - A)^{-1}B + D, \end{array}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ is called a realization of Σ.

Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad \text{with transfer function} \\ y(t) &= Cx(t) + Du(t), \quad G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ is called a realization of Σ.

Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Theorem

A realization (A, B, C, D) of a linear system is minimal $\Longleftrightarrow$
(A, B) is controllable and (A, C) is observable.

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1\text{)}.$$

When does a balanced realization exist?

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

When does a balanced realization exist?
Assume A to be Hurwitz, i.e. $\Lambda(A) \subset \mathbb{C}^-$. Then:

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U \Sigma V^T$ is the SVD of SR^T.

Definition

A realization (A, B, C, D) of a stable linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Definition

A realization (A, B, C, D) of a stable linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Theorem

The infinite controllability/observability Gramians P/Q satisfy the Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

Definition

A realization (A, B, C, D) of a stable linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Theorem

The infinite controllability/observability Gramians P/Q satisfy the Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

Proof. Exercise!

Definition

A realization (A, B, C, D) of a stable linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

Proof. In balanced coordinates, the HSVs are $\Lambda\left(PQ\right)^{\frac{1}{2}}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.$$

Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

Proof. In balanced coordinates, the HSVs are $\Lambda\left(PQ\right)^{\frac{1}{2}}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^T T^T + TBB^T T^T.$$

This is equivalent to

$$0 = A(T^{-1}\hat{P}T^{-T}) + (T^{-1}\hat{P}T^{-T})A^T + BB^T.$$

Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

Proof. In balanced coordinates, the HSVs are $\Lambda(PQ)^{\frac{1}{2}}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.$$

This is equivalent to

$$0 = A(T^{-1}\hat{P}T^{-T}) + (T^{-1}\hat{P}T^{-T})A^T + BB^T.$$

The uniqueness of the solution of the Lyapunov equation implies that $\hat{P} = TPT^T$ and, analogously, $\hat{Q} = T^{-T}QT^{-1}$. Therefore,

$$\hat{P}\hat{Q} = TPQT^{-1},$$

showing that $\Lambda(\hat{P}\hat{Q}) = \Lambda(PQ) = \{\sigma_1^2, \ldots, \sigma_n^2\}$.

Definition

A realization (A, B, C, D) of a stable linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\operatorname{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u^*(\jmath\omega)u(\jmath\omega)\,d\omega.$$

Assume A is (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \operatorname{re} z < 0\}$.

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(\jmath\omega)u(\jmath\omega)\,d\omega.$$

Assume A is (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C}\ :\ \mathrm{re}\, z < 0\}$.
Then for all $s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$, $\|G(s)\| \le M \le \infty \Rightarrow$

$$\int_{-\infty}^{\infty} y^*(\jmath\omega)y(\jmath\omega)\,d\omega \quad = \quad \int_{-\infty}^{\infty} u^*(\jmath\omega)G^*(\jmath\omega)G(\jmath\omega)u(\jmath\omega)\,d\omega$$

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u^*(\jmath\omega) u(\jmath\omega)\, d\omega.$$

Assume A is (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \operatorname{re} z < 0\}$.
Then for all $s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$, $\|G(s)\| \leq M \leq \infty \Rightarrow$

$$\begin{aligned}
\int_{-\infty}^{\infty} y^*(\jmath\omega) y(\jmath\omega)\, d\omega &= \int_{-\infty}^{\infty} u^*(\jmath\omega) G^*(\jmath\omega) G(\jmath\omega) u(\jmath\omega)\, d\omega \\
&= \int_{-\infty}^{\infty} \|G(\jmath\omega) u(\jmath\omega)\|^2\, d\omega \leq \int_{-\infty}^{\infty} M^2\, \|u(\jmath\omega)\|^2\, d\omega
\end{aligned}$$

(Here:, $\|\,.\,\|$ denotes the Euclidian vector or spectral matrix norm.)

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty,\infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(\jmath\omega)u(\jmath\omega)\,d\omega.$$

Assume A is (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C}\,:\,\operatorname{re} z < 0\}$.
Then for all $s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$, $\|G(s)\| \leq M \leq \infty \Rightarrow$

$$\begin{aligned}
\int_{-\infty}^{\infty} y^*(\jmath\omega)y(\jmath\omega)\,d\omega &= \int_{-\infty}^{\infty} u^*(\jmath\omega)G^*(\jmath\omega)G(\jmath\omega)u(\jmath\omega)\,d\omega \\
&= \int_{-\infty}^{\infty} \|G(\jmath\omega)u(\jmath\omega)\|^2\,d\omega \leq \int_{-\infty}^{\infty} M^2\,\|u(\jmath\omega)\|^2\,d\omega \\
&= M^2\int_{-\infty}^{\infty} u(\jmath\omega)^*u(\jmath\omega)\,d\omega \;<\; \infty.
\end{aligned}$$

(Here:, $\|\,.\,\|$ denotes the Euclidian vector or spectral matrix norm.)

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty,\infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(\jmath\omega)u(\jmath\omega)\,d\omega.$$

Assume A is (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \mathrm{re}\, z < 0\}$.
Then for all $s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$, $\|G(s)\| \leq M \leq \infty \Rightarrow$

$$\begin{aligned}
\int_{-\infty}^{\infty} y^*(\jmath\omega)y(\jmath\omega)\,d\omega &= \int_{-\infty}^{\infty} u^*(\jmath\omega)G^*(\jmath\omega)G(\jmath\omega)u(\jmath\omega)\,d\omega \\
&= \int_{-\infty}^{\infty} \|G(\jmath\omega)u(\jmath\omega)\|^2\;d\omega \leq \int_{-\infty}^{\infty} M^2\,\|u(\jmath\omega)\|^2\;d\omega \\
&= M^2\int_{-\infty}^{\infty} u(\jmath\omega)^*u(\jmath\omega)\,d\omega \;<\; \infty.
\end{aligned}$$

$\Longrightarrow y \in L_2^p(-\infty,\infty) \cong \mathcal{L}_2^p.$

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(\jmath\omega)u(\jmath\omega)\,d\omega.$$

Assume A is (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \operatorname{re} z < 0\}$.

Consequently, the 2-induced operator norm

$$\|G\|_\infty := \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2}$$

is well defined. It can be shown that

$$\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(G(\jmath\omega)\right).$$

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u^*(\jmath\omega)u(\jmath\omega)\,d\omega.$$

Assume A is (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \operatorname{re} z < 0\}$.

Hardy space $\mathcal{H}_\infty$

Function space of analytic and bounded (in $\mathbb{C}^+$) matrix-/scalar-valued functions. The $\mathcal{H}_\infty$-norm is

$$\|F\|_\infty := \sup_{\operatorname{re} s>0} \sigma_{\max}\,(F(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{max}\,(F(\jmath\omega))\,.$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = p = 1$);
- $\mathcal{H}_\infty^{p\times m}$ in the MIMO case (multi-input, multi-output, $m > 1, p > 1$).

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D.$$

Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty,\infty) \cong \mathcal{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D.$$

Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty,\infty) \cong \mathcal{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_2 = \left\|Gu - \hat{G}u\right\|_2 \leq \left\|G - \hat{G}\right\|_\infty \|u\|_2\,.$$

$\Longrightarrow$ compute reduced-order model such that $\left\|G - \hat{G}\right\|_\infty < tol$!

Note: error bound holds in time- and frequency domain due to Paley-Wiener!

Consider transfer function $G(s) = C\,(sI - A)^{-1}\,B,$ i.e. $D = 0$.

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic in $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\begin{aligned}\|F\|_2 \quad &:= \quad \left(\sup_{\operatorname{re}\sigma>0} \int_{-\infty}^{\infty} \|F(\sigma + \jmath\omega)\|_F^2 \; d\omega\right)^{\frac{1}{2}} \\ &= \quad \left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2 \; d\omega\right)^{\frac{1}{2}}.\end{aligned}$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_2$ in the SISO case (single-input, single-output, $m = p = 1$);
- $\mathcal{H}_2^{p\times m}$ in the MIMO case (multi-input, multi-output, $m > 1, p > 1$).

Consider transfer function $G(s) = C\,(sI - A)^{-1}\,B$, i.e. $D = 0$.

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic in $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\|F\|_2 \quad = \quad \left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2 \, d\omega\right)^{\frac{1}{2}}.$$

$\mathcal{H}_2$ approximation error for impulse response $(u(t) = u_0\delta(t))$

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_2 = \left\|Gu_0\delta - \hat{G}\,u_0\delta\right\|_2 \leq \left\|G - \hat{G}\right\|_2 \|u_0\|.$$

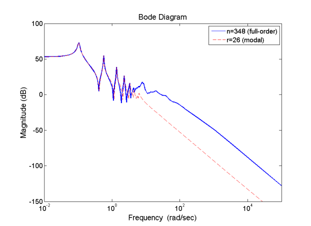
$\Longrightarrow$ compute reduced-order model such that $\left\|G - \hat{G}\right\|_2 < tol$!

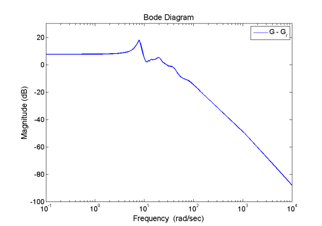
$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory)

Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\left\| G(\jmath\omega_j) - \hat{G}(\jmath\omega_j) \right\|_2$, $\left\| G(\jmath\omega_j) - \hat{G}(\jmath\omega_j) \right\|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\left\| G(\jmath\omega_j) - \hat{G}(\jmath\omega_j) \right\|_2}{\| G(\jmath\omega_j) \|_2}$, $\frac{\left\| G(\jmath\omega_j) - \hat{G}(\jmath\omega_j) \right\|_\infty}{\| G(\jmath\omega_j) \|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot:
  - for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20 \log_{10}$(value);
  - for MIMO systems, $p \times m$ array of of plots G_{ij}.





Outline

- Automatic generation of compact models.

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

Projector

A projector is a matrix $P \in \mathbb{R}^{n\times n}$ with $P^2 = P$. Let $\mathcal{V} = \operatorname{range}(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Projector

A projector is a matrix $P \in \mathbb{R}^{n\times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r\,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Properties:

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector. (aka: Petrov-Galerkin projection.)

Projector

A projector is a matrix $P \in \mathbb{R}^{n\times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^TV)^{-1}V^T$ is a projector onto $\mathcal{V}$.

Properties:

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector. (aka: Petrov-Galerkin projection.)
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.

Projector

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \text{range}(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Properties:

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector. (aka: Petrov-Galerkin projection.)
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.

Projector

A projector is a matrix $P \in \mathbb{R}^{n\times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^TV)^{-1}V^T$ is a projector onto $\mathcal{V}$.

Properties:

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector. (aka: Petrov-Galerkin projection.)
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.

Projector

A projector is a matrix $P \in \mathbb{R}^{n\times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Properties:

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector. (aka: Petrov-Galerkin projection.)
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\, w_1, \ldots, w_r \,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Model Reduction by Projection

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods) $\rightsquigarrow$ Part II of tutorial, by Serkan Gugercin!
3. Balanced Truncation
4. many more...

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, where

$$\operatorname{range}(V) = \mathcal{V}, \quad \operatorname{range}(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!
Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, and the reduced-order model is $\hat{x} = W^T x$

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T\left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) \quad = \quad W^T\left(VW^T\dot{x} - AVW^T x - Bu\right)$$

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^Tx =: \tilde{x}$, and the reduced-order model is $\hat{x} = W^Tx$

$$\hat{A} := W^TAV, \quad \hat{B} := W^TB, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$\begin{aligned} W^T\left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) &= W^T\left(VW^T\dot{x} - AVW^Tx - Bu\right) \\ &= \underbrace{W^T\dot{x}}_{\dot{\hat{x}}} - \underbrace{W^TAV}_{=\hat{A}}\underbrace{W^Tx}_{=\hat{x}} - \underbrace{W^TB}_{=\hat{B}}u \end{aligned}$$

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, and the reduced-order model is $\hat{x} = W^T x$

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$\begin{aligned} W^T\left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) &= W^T\left(VW^T\dot{x} - AVW^T x - Bu\right) \\ &= \underbrace{W^T\dot{x}}_{\dot{\hat{x}}} - \underbrace{W^T AV}_{=\hat{A}}\underbrace{W^T x}_{=\hat{x}} - \underbrace{W^T B}_{=\hat{B}}u \\ &= \dot{\hat{x}} - \hat{A}\hat{x} - \hat{B}u = 0. \end{aligned}$$

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) \quad = \quad \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right)$$

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\ &= C\left((sI_n - A)^{-1} - V(sI_r - \hat{A})^{-1}W^T\right)B \end{aligned}$$

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\ &= C\left((sI_n - A)^{-1} - V(sI_r - \hat{A})^{-1}W^T\right)B \\ &= C\left(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\right)(sI_n - A)^{-1}B. \end{aligned}$$

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\ &= C\big(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\big)(sI_n - A)^{-1}B. \end{aligned}$$

$P(s)$ is a projector onto $\mathcal{V}$:

range$(P(s)) \subset$ range(V), all matrices have full rank $\Rightarrow$ "$=$", and

$$P(s)^2 = V(sI_r - \hat{A})^{-1}W^T(sI_n - A)V(sI_r - \hat{A})^{-1}W^T(sI_n - A)$$

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\ &= C\left(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\right)(sI_n - A)^{-1}B. \end{aligned}$$

$P(s)$ is a projector onto $\mathcal{V}$:

range$(P(s)) \subset$ range(V), all matrices have full rank $\Rightarrow$ "=", and

$$\begin{aligned} P(s)^2 &= V(sI_r - \hat{A})^{-1}W^T(sI_n - A)V(sI_r - \hat{A})^{-1}W^T(sI_n - A) \\ &= V(sI_r - \hat{A})^{-1}\underbrace{(sI_r - \hat{A})(sI_r - \hat{A})^{-1}}_{=I_r}W^T(sI_n - A) = P(s). \end{aligned}$$

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\ &= C\left(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\right)(sI_n - A)^{-1}B. \end{aligned}$$

$P(s)$ is a projector onto $\mathcal{V}$ $\Longrightarrow$

Given $s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A})\right)$,

if $(s_* I_n - A)^{-1}B \in \mathcal{V}$, *then* $(I_n - P(s_*))(s_* I_n - A)^{-1}B = 0$,

hence $G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*)$, i.e., $\hat{G}$ interpolates G in s_*!

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\ &= C\Big(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\Big)(sI_n - A)^{-1}B. \\ \text{Analogously,} \quad &= C(sI_n - A)^{-1}\Big(I_n - \underbrace{(sI_n - A)V(sI_r - \hat{A})^{-1}W^T}_{=:Q(s)}\Big)B. \end{aligned}$$

$Q(s)^*$ is a projector onto $\mathcal{W}$ $\Longrightarrow$ Given $s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A})\right)$,

if $(s_* I_n - A)^{-T} C^T \in \mathcal{W}$, *then* $C(s_* I_n - A)^{-1}(I_n - Q(s_*)) = 0$,

hence $G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*)$, i.e., $\hat{G}$ interpolates G in s_*!

Theorem [Grimme 1997, Villemagne/Skelton 1987]

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

and $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, if either

- $(s_* I_n - A)^{-1} B \in \operatorname{range}(V)$, or
- $(s_* I_n - A)^{-T} C^T \in \operatorname{range}(W)$,

then at $s = s_*$, we obtain the (rational) interpolation condition

$$G(s_*) = \hat{G}(s_*).$$

Note: extension to Hermite interpolation $\rightsquigarrow$ Part II!

Base enrichment

Static modes are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \quad \Longrightarrow \quad x(t) \equiv -A^{-1} b_j.$$

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[\, b_1, \ldots, b_m \,] = A^{-1} B$.

Interpolation-projection framework $\Longrightarrow G(0) = \hat{G}(0)$!

If two-sided projection is used, complimentary subspace can be augmented by $A^{-T} C^T \Longrightarrow G'(0) = \hat{G}'(0)$!

Note: if $m \neq q$, add random vectors or delete some of the columns in $A^{-T} C^T$.

Guyan reduction (static condensation)

Partition states in masters $x_1 \in \mathbb{R}^r$ and slaves $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)
Assume stationarity, i.e., $\dot{x} = 0$ and solve for x_2 in

$$
\begin{aligned}
0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\
\Rightarrow \quad x_2 &= -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u.
\end{aligned}
$$

Inserting this into the first part of the dynamic system

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u, \quad y = C_1 x_1 + C_2 x_2$$

then yields the reduced-order model

$$
\begin{aligned}
\dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2) u \\
y &= (C_1 - C_2 A_{22}^{-1} A_{21}) x_1 - C_2 A_{22}^{-1} B_2 u.
\end{aligned}
$$

Outline

1. Introduction

2. Model Reduction by Projection

3. Balanced Truncation
   The basic method
   ADI Methods for Lyapunov Equations
   Balancing-Related Model Reduction

4. Final Remarks

Balanced Truncation

Basic principle:

- Recall: an LTI system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

Basic principle:

- Recall: an LTI system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T \;=\; 0, \qquad A^T Q + QA + C^T C \;=\; 0,$$

  satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Balanced Truncation

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC \; = \; 0.$
- $\Lambda\left(PQ\right)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\, d\tau$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \ = \ 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau =: Ce^{At}\underbrace{\int_{-\infty}^{0} e^{-A\tau}Bu(\tau)\,d\tau}_{=:z}$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC \ = \ 0.$
- $\Lambda\left(PQ\right)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau =: Ce^{At}\underbrace{\int_{-\infty}^{0} e^{-A\tau}Bu(\tau)\,d\tau}_{=:z} = Ce^{At}z.$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC \;=\; 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau = Ce^{At}z.$$

  Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC \;=\; 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau = Ce^{At}z.$$

Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*y(t) = \int_0^{\infty} B^Te^{A^T(\tau-t)}C^Ty(\tau)\,d\tau$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \; = \; 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau = Ce^{At}z.$$

  Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*y(t) = \int_0^\infty B^T e^{A^T(\tau-t)}C^T y(\tau)\,d\tau = B^T e^{-A^T t}\int_0^\infty e^{A^T\tau}C^T y(\tau)\,d\tau.$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \ = \ 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau = Ce^{At}z.$$

Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*y(t) = \ = B^T e^{-A^T t}\int_0^{\infty} e^{A^T\tau}C^T y(\tau)\,d\tau.$$

Hence,

$$\mathcal{H}^*\mathcal{H}u(t) \quad = \quad B^T e^{-A^T t}\int_0^{\infty} e^{A^T\tau}C^T Ce^{A\tau}z\,d\tau$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \;=\; 0.$
- $\Lambda\left(PQ\right)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau = Ce^{At}z.$$

Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^* y(t) = \;= B^T e^{-A^T t} \int_0^\infty e^{A^T \tau} C^T y(\tau)\,d\tau.$$

Hence,

$$\begin{aligned} \mathcal{H}^*\mathcal{H}u(t) &= B^T e^{-A^T t} \int_0^\infty e^{A^T \tau} C^T C e^{A\tau} z\,d\tau \\ &= B^T e^{-A^T t} \underbrace{\int_0^\infty e^{A^T \tau} C^T C e^{A\tau}\,d\tau}_{\equiv Q}\; z \end{aligned}$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \ = \ 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau)\, d\tau = Ce^{At}z.$$

Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^* y(t) = \ = B^T e^{-A^T t} \int_0^{\infty} e^{A^T\tau} C^T y(\tau)\, d\tau.$$

Hence,

$$\begin{aligned} \mathcal{H}^*\mathcal{H}u(t) &= B^T e^{-A^T t} \int_0^{\infty} e^{A^T\tau} C^T Ce^{A\tau} z\, d\tau \\ &= B^T e^{-A^T t} Qz \end{aligned}$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \;=\; 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

  $$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau = Ce^{At}z.$$

  Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

  $$\mathcal{H}^*y(t) = \;= B^T e^{-A^T t} \int_0^{\infty} e^{A^T \tau} C^T y(\tau)\,d\tau.$$

  Hence,

  $$\mathcal{H}^*\mathcal{H}u(t) \;=\; B^T e^{-A^T t} Qz$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \ = \ 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau = Ce^{At}z.$$

Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^* y(t) = \ = B^T e^{-A^T t} \int_0^{\infty} e^{A^T \tau} C^T y(\tau)\,d\tau.$$

Hence,

$$\mathcal{H}^*\mathcal{H}u(t) \ = \ B^T e^{-A^T t} Qz \ \doteq \ \sigma^2 u(t).$$

Balanced Truncation

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC \; = \; 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*\mathcal{H}u(t) \;\; = \;\; B^T e^{-A^T t} Qz \; \doteq \; \sigma^2 u(t).$$

$\Longrightarrow u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC \; = \; 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Proof: Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*\mathcal{H}u(t) \;\; = \;\; B^T e^{-A^T t} Qz \; \doteq \; \sigma^2 u(t).$$

$\Longrightarrow u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz \Longrightarrow$ (recalling $z = \int_{-\infty}^{0} e^{-A\tau} Bu(\tau)\, d\tau$)

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC \ = \ 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Proof: Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*\mathcal{H}u(t) \quad = \quad B^Te^{-A^Tt}Qz \ \doteq \ \sigma^2 u(t).$$

$$\Longrightarrow u(t) = \tfrac{1}{\sigma^2}B^Te^{-A^Tt}Qz \Longrightarrow (\text{recalling } z = \textstyle\int_{-\infty}^0 e^{-A\tau}Bu(\tau)\,d\tau)$$

$$z \quad = \quad \int_{-\infty}^0 e^{-A\tau}B\frac{1}{\sigma^2}B^Te^{-A^T\tau}Qz\,d\tau$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \;=\; 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

  Proof: Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*\mathcal{H}u(t) \;=\; B^T e^{-A^T t} Qz \doteq \sigma^2 u(t).$$

$$\Longrightarrow u(t) = \tfrac{1}{\sigma^2} B^T e^{-A^T t} Qz \Longrightarrow (\text{recalling } z = \textstyle\int_{-\infty}^{0} e^{-A\tau} B u(\tau)\, d\tau)$$

$$\begin{aligned} z &= \int_{-\infty}^{0} e^{-A\tau} B \frac{1}{\sigma^2} B^T e^{-A^T \tau} Qz\, d\tau \\ &= \frac{1}{\sigma^2} \int_{-\infty}^{0} e^{-A\tau} BB^T e^{-A^T \tau}\, d\tau\, Qz \end{aligned}$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \ = \ 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Proof: Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*\mathcal{H}u(t) \quad = \quad B^T e^{-A^T t} Qz \doteq \sigma^2 u(t).$$

$$\Longrightarrow u(t) = \tfrac{1}{\sigma^2} B^T e^{-A^T t} Qz \Longrightarrow (\text{recalling } z = \textstyle\int_{-\infty}^0 e^{-A\tau} Bu(\tau)\, d\tau)$$

$$\begin{aligned} z &= \int_{-\infty}^0 e^{-A\tau} B \frac{1}{\sigma^2} B^T e^{-A^T\tau} Qz\, d\tau \\ &= \frac{1}{\sigma^2} \int_{-\infty}^0 e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau\, Qz \\ &= \frac{1}{\sigma^2} \underbrace{\int_0^\infty e^{At} BB^T e^{A^T t}\, dt}_{\equiv P}\, Qz \end{aligned}$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \ = \ 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Proof: Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*\mathcal{H}u(t) \quad = \quad B^T e^{-A^T t} Qz \ \doteq \ \sigma^2 u(t).$$

$\Longrightarrow u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz \Longrightarrow$ (recalling $z = \int_{-\infty}^{0} e^{-A\tau} Bu(\tau)\, d\tau$)

$$\begin{aligned} z &= \int_{-\infty}^{0} e^{-A\tau} B \frac{1}{\sigma^2} B^T e^{-A^T\tau} Qz \, d\tau \\ &= \frac{1}{\sigma^2} \underbrace{\int_{0}^{\infty} e^{At} BB^T e^{A^T t}\, dt}_{\equiv P} \, Qz \\ &= \frac{1}{\sigma^2} PQz \end{aligned}$$

Basic principle:

- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C \ = \ 0.$
- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Proof: Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^*\mathcal{H}u(t) \quad = \quad B^T e^{-A^T t} Qz \ \doteq \ \sigma^2 u(t).$$

$\Longrightarrow u(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz \Longrightarrow$ (recalling $z = \int_{-\infty}^0 e^{-A\tau} B u(\tau)\, d\tau$)

$$\begin{aligned} z \quad &= \quad \int_{-\infty}^0 e^{-A\tau} B \frac{1}{\sigma^2} B^T e^{-A^T\tau} Qz\, d\tau \\ &= \quad \frac{1}{\sigma^2} \underbrace{\int_0^\infty e^{At} BB^T e^{A^T t}\, dt}_{\equiv P} Qz \\ &= \quad \frac{1}{\sigma^2} PQz \end{aligned}$$

$\Longleftrightarrow \qquad PQz = \sigma^2 z. \quad \square$

Basic principle:

- Recall: an LTI system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\Lambda (PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

Basic principle:

- Recall: an LTI system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

Motivation:

HSVs are system invariants: they are preserved under
$\mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$:

in transformed coordinates, the Gramians satisfy

$$\begin{aligned}(TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0,\\ (TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0\end{aligned}$$

$$\Rightarrow\ (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},$$

hence $\Lambda(PQ) = \Lambda((TPT^T)(T^{-T}QT^{-1}))$.

Motivation:

HSVs are system invariants: they are preserved under
$\mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$.

HSVs determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

In balanced coordinates ... energy transfer from u_- to y_+:

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\,dt}{\int_{-\infty}^0 u(t)^T u(t)\,dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^{n} \sigma_j^2 x_{0,j}^2$$

Motivation:

HSVs are system invariants: they are preserved under
$\mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$.

HSVs determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

In balanced coordinates ... energy transfer from u_- to y_+:

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int\limits_0^\infty y(t)^T y(t)\, dt}{\int\limits_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

$\Longrightarrow$ Truncate states corresponding to “small” HSVs

$\Longrightarrow$ complete analogy to best approximation via SVD!

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S,\ Q = R^T R.$

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S, \; Q = R^T R$.
2. Compute SVD $SR^T = [\, U_1, \; U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S,\; Q = R^T R$.
2. Compute SVD $SR^T = [\, U_1,\, U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S, \; Q = R^T R$.
2. Compute SVD $SR^T = [\, U_1, \; U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

Note:

$$V^T W \;=\; (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}})$$

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S,\ Q = R^T R$.
2. Compute SVD $SR^T = [\,U_1,\ U_2\,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

Note:

$$V^T W \;=\; (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}}) \;=\; \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}}$$

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S, \; Q = R^T R$.
2. Compute SVD $SR^T = [\, U_1, \, U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

Note:

$$\begin{aligned} V^T W &= (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}}) = \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}} \\ &= \Sigma_1^{-\frac{1}{2}} [\, I_r, \, 0 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \Sigma_1^{-\frac{1}{2}} \end{aligned}$$

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S,\; Q = R^T R$.
2. Compute SVD $SR^T = [\, U_1,\, U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

Note:

$$\begin{aligned} V^T W &= (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}}) = \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}} \\ &= \Sigma_1^{-\frac{1}{2}} [\, I_r,\, 0 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \Sigma_1^{-\frac{1}{2}} = \Sigma_1^{-\frac{1}{2}} \Sigma_1 \Sigma_1^{-\frac{1}{2}} = I_r \end{aligned}$$

$\Longrightarrow$ VW^T is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2 .$$

Balanced Truncation

Properties:

General misconception: complexity $\mathcal{O}(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

Balanced Truncation

Properties:

General misconception: complexity $\mathcal{O}(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

Use low-rank techniques ideas from numerical linear algebra:

Properties:

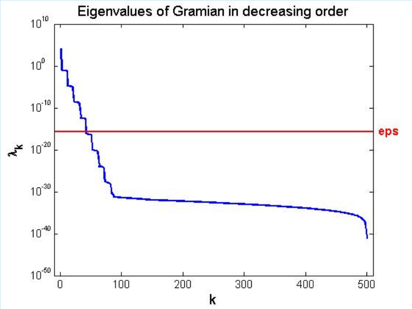
General misconception: complexity $\mathcal{O}(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

Use low-rank techniques ideas from numerical linear algebra:

- Instead of Gramians P, Q compute $S, R \in \mathbb{R}^{n \times k}$, $k \ll n$, such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

- Compute S, R with problem-specific Lyapunov solvers of "low" complexity directly.



Properties:

General misconception: complexity $\mathcal{O}(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

Use low-rank techniques ideas from numerical linear algebra:

Sparse Balanced Truncation:

- Implementation using sparse Lyapunov solver ($\rightarrow$ ADI+sparse LU).
- Complexity $\mathcal{O}(n(k^2 + r^2))$.
- Software:
  - \+ MATLAB toolbox LyaPack (Penzl 1999),
  - \+ Software library M.E.S.S.[a] in C/MATLAB [B./Saak/Köhler/uvm.],
  - \+ pyMOR.

[a] Matrix Equation Sparse Solvers

Recall Peaceman-Rachford ADI:

Consider $Au = s$ where $A \in \mathbb{R}^{n \times n}$ spd, $s \in \mathbb{R}^n$.

ADI iteration idea: decompose $A = H + V$ with $H, V \in \mathbb{R}^{n \times n}$ such that

$$\begin{aligned}(H + pI)v &= r \\ (V + pI)w &= t\end{aligned}$$

can be solved easily/efficiently.

Recall Peaceman-Rachford ADI:

Consider $Au = s$ where $A \in \mathbb{R}^{n \times n}$ spd, $s \in \mathbb{R}^n$.

ADI iteration idea: decompose $A = H + V$ with $H, V \in \mathbb{R}^{n \times n}$ such that

$$\begin{aligned}(H + pI)v &= r\\(V + pI)w &= t\end{aligned}$$

can be solved easily/efficiently.

ADI Iteration

If H, V spd $\Rightarrow \exists p_k,\ k = 1, 2, \ldots,$ such that

$$\begin{aligned}u_0 &= 0\\(H + p_k I)u_{k-\frac{1}{2}} &= (p_k I - V)u_{k-1} + s\\(V + p_k I)u_k &= (p_k I - H)u_{k-\frac{1}{2}} + s\end{aligned}$$

converges to $u \in \mathbb{R}^n$ solving $Au = s$.

ADI Methods for Lyapunov Equations

The Lyapunov operator

$$\mathcal{L}: \quad P \quad \mapsto \quad AX + XA^T$$

can be decomposed into the linear operators

$$\mathcal{L}_H : X \mapsto AX, \qquad \mathcal{L}_V : X \mapsto XA^T.$$

In analogy to the standard ADI method we find the

ADI iteration for the Lyapunov equation **[Wachspress 1988]**

$$\begin{aligned} X_0 &= 0, \\ (A + p_k I)X_{k-\frac{1}{2}} &= -W - X_{k-1}(A^T - p_k I), \\ (A + p_k I)X_k^T &= -W - X_{k-\frac{1}{2}}^T(A^T - p_k I). \end{aligned}$$

Consider $AX + XA^T = -BB^T$ for stable A, $B \in \mathbb{R}^{n \times m}$ with $m \ll n$.

ADI iteration for the Lyapunov equation [Wachspress 1988]

For $k = 1, \ldots, k_{\max}$

$$\begin{aligned} X_0 &= 0 \\ (A + p_k I) X_{k-\frac{1}{2}} &= -BB^T - X_{k-1}(A^T - p_k I) \\ (A + p_k I) X_k^T &= -BB^T - X_{k-\frac{1}{2}}^T (A^T - p_k I) \end{aligned}$$

Consider $AX + XA^T = -BB^T$ for stable A, $B \in \mathbb{R}^{n\times m}$ with $m \ll n$.

ADI iteration for the Lyapunov equation [Wachspress 1988]

For $k = 1, \ldots, k_{\max}$

$$\begin{aligned} X_0 &= 0 \\ (A + p_k I)X_{k-\frac{1}{2}} &= -BB^T - X_{k-1}(A^T - p_k I) \\ (A + p_k I)X_k^T &= -BB^T - X_{k-\frac{1}{2}}^T(A^T - p_k I) \end{aligned}$$

Rewrite as one step iteration and factorize $X_k = Z_k Z_k^T$, $k = 0, \ldots, k_{\max}$

$$\begin{aligned} Z_0 Z_0^T &= 0 \\ Z_k Z_k^T &= -2p_k (A + p_k I)^{-1} BB^T (A + p_k I)^{-T} \\ &\quad + (A + p_k I)^{-1}(A - p_k I) Z_{k-1} Z_{k-1}^T (A - p_k I)^T (A + p_k I)^{-T} \end{aligned}$$

Consider $AX + XA^T = -BB^T$ for stable A, $B \in \mathbb{R}^{n \times m}$ with $m \ll n$.

ADI iteration for the Lyapunov equation [Wachspress 1988]

For $k = 1, \ldots, k_{\max}$

$$\begin{aligned} X_0 &= 0 \\ (A + p_k I)X_{k-\frac{1}{2}} &= -BB^T - X_{k-1}(A^T - p_k I) \\ (A + p_k I)X_k^T &= -BB^T - X_{k-\frac{1}{2}}^T(A^T - p_k I) \end{aligned}$$

Rewrite as one step iteration and factorize $X_k = Z_k Z_k^T$, $k = 0, \ldots, k_{\max}$

$$\begin{aligned} Z_0 Z_0^T &= 0 \\ Z_k Z_k^T &= -2p_k(A + p_k I)^{-1}BB^T(A + p_k I)^{-T} \\ &\quad + (A + p_k I)^{-1}(A - p_k I)Z_{k-1}Z_{k-1}^T(A - p_k I)^T(A + p_k I)^{-T} \end{aligned}$$

$\ldots \rightsquigarrow$ low-rank Cholesky factor ADI [Penzl 1997/2000, Li/White 1999/2002, B./Li/Penzl 1999/2008, Gugercin/Sorensen/Antoulas 2003]

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B,\ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$ [Penzl '00]

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B,\ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$ [Penzl '00]

Observing that $(A - p_i I)$, $(A + p_k I)^{-1}$ commute, we rewrite $Z_{k_{\max}}$ as

$$Z_{k_{\max}} = [z_{k_{\max}},\ P_{k_{\max}-1}z_{k_{\max}},\ P_{k_{\max}-2}(P_{k_{\max}-1}z_{k_{\max}}),\ \ldots,\ P_1(P_2 \cdots P_{k_{\max}-1}z_{k_{\max}})],$$

where

$$z_{k_{\max}} = \sqrt{-2p_{k_{\max}}}(A + p_{k_{\max}} I)^{-1}B$$

and

$$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}}\left[I - (p_i + p_{i+1})(A + p_i I)^{-1}\right].$$

[Li/White '02]

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B,\ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$ [Penzl '00]

Observing that $(A - p_i I)$, $(A + p_k I)^{-1}$ commute, we rewrite $Z_{k_{\max}}$ as

$$Z_{k_{\max}} = [z_{k_{\max}},\ P_{k_{\max}-1}z_{k_{\max}},\ P_{k_{\max}-2}(P_{k_{\max}-1}z_{k_{\max}}),\ \ldots,\ P_1(P_2 \cdots P_{k_{\max}-1}z_{k_{\max}})],$$

where

$$z_{k_{\max}} = \sqrt{-2p_{k_{\max}}}(A + p_{k_{\max}} I)^{-1}B$$

and

$$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}}\left[I - (p_i + p_{i+1})(A + p_i I)^{-1}\right].$$

[Li/White '02]

$\rightsquigarrow$ Need to solve only one (sparse) linear system with m right-hand sides per iteration!

Algorithm [Penzl 1997/2000, Li/White 1999/2002, B. 2004, B./Li/Penzl 1999/2008]

$$V_1 \quad \leftarrow \quad \sqrt{-2\,\mathrm{re}\,p_1}(A + p_1 I)^{-1} B, \qquad Z_1 \;\leftarrow\; V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{re}\,p_k}{\mathrm{re}\,p_{k-1}}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right)$$

$$Z_k \leftarrow \begin{bmatrix} Z_{k-1} & V_k \end{bmatrix}$$

$$Z_k \leftarrow \mathrm{rrlq}(Z_k, \tau) \qquad \text{% column compression, optional}$$

Algorithm **[Penzl 1997/2000, Li/White 1999/2002, B. 2004, B./Li/Penzl 1999/2008]**

$$V_1 \leftarrow \sqrt{-2\,\mathrm{re}\,p_1}(A + p_1 I)^{-1}B, \qquad Z_1 \leftarrow V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{re}\,p_k}{\mathrm{re}\,p_{k-1}}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1}V_{k-1}\right)$$

$$Z_k \leftarrow \begin{bmatrix} Z_{k-1} & V_k \end{bmatrix}$$

$$Z_k \leftarrow \mathrm{rrlq}(Z_k, \tau) \qquad \text{\% column compression, optional}$$

At convergence, $Z_{k_{\max}} Z_{k_{\max}}^T \approx X$, where (without column compression)

$$Z_{k_{\max}} = \begin{bmatrix} V_1 & \ldots & V_{k_{\max}} \end{bmatrix}, \qquad V_k = \Box \in \mathbb{C}^{n \times m}.$$

Algorithm **[Penzl 1997/2000, Li/White 1999/2002, B. 2004, B./Li/Penzl 1999/2008]**

$$V_1 \leftarrow \sqrt{-2\,\mathrm{re}\, p_1}(A + p_1 I)^{-1} B, \qquad Z_1 \leftarrow V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{re}\, p_k}{\mathrm{re}\, p_{k-1}}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right)$$

$$Z_k \leftarrow \left[\begin{array}{cc} Z_{k-1} & V_k \end{array}\right]$$

$$Z_k \leftarrow \mathrm{rrlq}(Z_k, \tau) \qquad \text{\% column compression, optional}$$

At convergence, $Z_{k_{\max}} Z_{k_{\max}}^T \approx X$, where (without column compression)

$$Z_{k_{\max}} = \left[\begin{array}{ccc} V_1 & \ldots & V_{k_{\max}} \end{array}\right], \quad V_k = \boxed{} \in \mathbb{C}^{n \times m}.$$

Note: Implementation in real arithmetic is possible: combine two steps [B./Li/Penzl 1999/2008] or employ the relations of consecutive complex factors [B./Kürschner/Saak 2011].

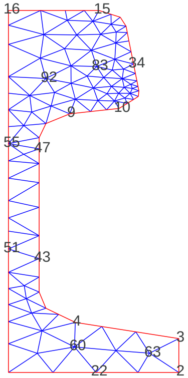
Current implementations (pyMOR, M.E.S.S.) employ low-rank property of residual, update residual in each step, and compute new shifts on the fly!

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, && \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa (u_k - x), && \xi \in \Gamma_k,\ 1 \le k \le 7, \\
\frac{\partial}{\partial n} x &= 0, && \xi \in \Gamma_7.
\end{aligned}
$$

$\Longrightarrow m = 7, p = 6.$

- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841.$

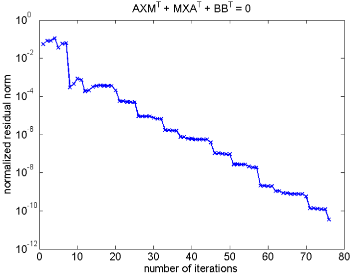


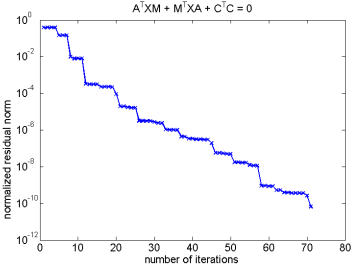
Source: Physical model: courtesy of Mannesmann/Demag.
Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

- Solve dual Lyapunov equations needed for balanced truncation, i.e.,

$$APM^T + MPA^T + BB^T = 0, \quad A^TQM + M^TQA + C^TC = 0,$$

for $79,841$.
- 25 shifts chosen by Penzl heuristic from 50/25 Ritz values of A of largest/smallest magnitude, no column compression performed.
- M.E.S.S. requires no factorization of mass matrix.
- Computations done on Core2Duo at 2.8GHz with 3GB RAM and 32Bit-MATLAB.





Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis range(Z), $Z \in \mathbb{R}^{n \times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T A Z$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$\mathcal{Z} = \mathcal{K}(A, B, r) = \mathrm{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}$$

[Saad 1990, Jaimoukha/Kasenally 1994, Jbilou 2002–08].

Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis range(Z), $Z \in \mathbb{R}^{n\times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$\mathcal{Z} = \mathcal{K}(A, B, r) = \mathrm{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}$$

[Saad 1990, Jaimoukha/Kasenally 1994, Jbilou 2002–08].

- Extended (and rational) Krylov method (EKSM, RKSM) [Simoncini 2007, Druskin/Knizhnerman/Simoncini 2011],

$$\mathcal{Z} = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).$$

Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis range(Z), $Z \in \mathbb{R}^{n \times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T A Z$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

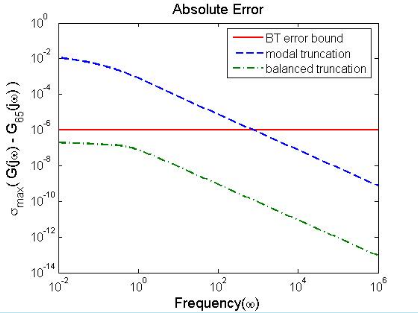
- ADI subspace [B./R.-C. Li/Truhar 2008]:

$$\mathcal{Z} = \mathrm{colspan}\left[\begin{array}{ccc} V_1, & \ldots, & V_r \end{array}\right].$$

Note:

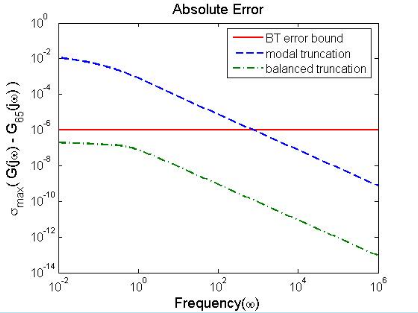
1. ADI subspace is rational Krylov subspace [J.-R. Li/White 2002].
2. Similar approach: ADI-preconditioned global Arnoldi method [Jbilou 2008].

$n = 1357$, Absolute Error



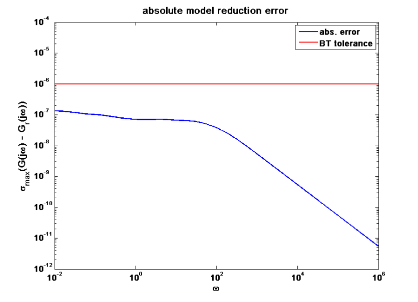
- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

$n = 1357$, Absolute Error

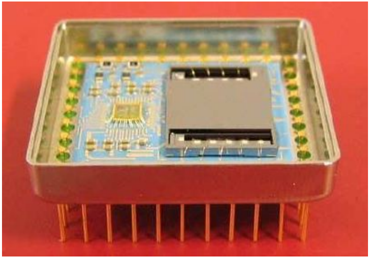


- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

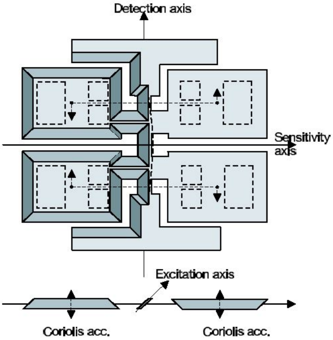
$n = 79841$, Absolute Error



- BT model computed using M.E.S.S. in MATLAB,
- dualcore, computation time: <10 min.



- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.



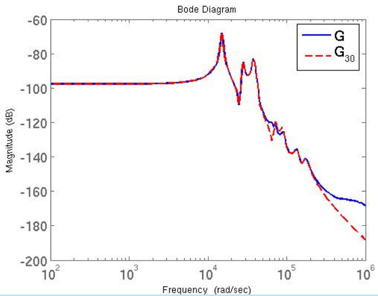
- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

Source: http://modelreduction.org/index.php/Modified_Gyroscope

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722$, $m = 1$, $p = 12$.
- Reduced model computed using SpaRed, $r = 30$.

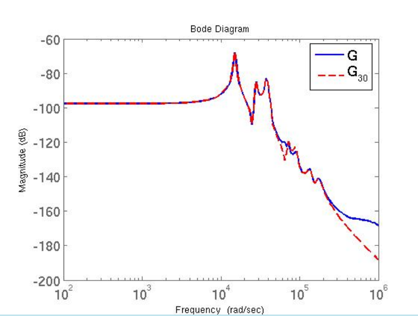
- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722$, $m = 1$, $p = 12$.
- Reduced model computed using SpaRed, $r = 30$.

Frequency Repsonse Analysis

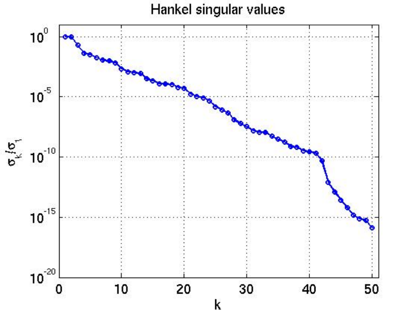


- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722$, $m = 1$, $p = 12$.
- Reduced model computed using SpaRed, $r = 30$.

Frequency Repsonse Analysis



Hankel Singular Values



Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Classical Balanced Truncation (BT) [MULLIS/ROBERTS 1976, MOORE 1981]

- $P =$ controllability Gramian of system given by (A, B, C, D).
- $Q =$ observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

LQG Balanced Truncation (LQGBT) [Jonckheere/Silverman 1983]

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Balanced Stochastic Truncation (BST) [Desai/Pal 1984, Green 1988]

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT) [Green '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^T Q + Q\bar{A} + QB\bar{R}^{-1}B^T Q + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

- Guaranteed preservation of physical properties like

- Guaranteed preservation of physical properties like
  - stability (all),

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).
- Computable error bounds, e.g.,

$$
\begin{aligned}
\text{BT:} \quad \|G - G_r\|_\infty &\leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT}, \\
\text{LQGBT:} \quad \|G - G_r\|_\infty &\leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}} \\
\text{BST:} \quad \|G - G_r\|_\infty &\leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty ,
\end{aligned}
$$

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).
- Computable error bounds, e.g.,

$$\begin{aligned}
\text{BT:} \quad \|G - G_r\|_\infty &\leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT}, \\
\text{LQGBT:} \quad \|G - G_r\|_\infty &\leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}} \\
\text{BST:} \quad \|G - G_r\|_\infty &\leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty,
\end{aligned}$$

- Can be combined with singular perturbation approximation (= Guyan reduction applied to balanced realization!) for improved steady-state performance.

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).
- Computable error bounds, e.g.,

$$\begin{aligned}
\text{BT:} \quad & \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT}, \\
\text{LQGBT:} \quad & \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}} \\
\text{BST:} \quad & \|G - G_r\|_\infty \leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty ,
\end{aligned}$$

- Can be combined with singular perturbation approximation (= Guyan reduction applied to balanced realization!) for improved steady-state performance.
- Computations can be modularized $\rightsquigarrow$ software packages M-M.E.S.S., MORLAB, see http://www.mpi-magdeburg.mpg.de/823508/software.

Outline

Current Research Topics

- Special methods for second-order (mechanical) and delay systems.
- Extensions to bilinear, quadratic-bilinear, polynomial, and stochastic systems.
- Empirical variants using snapshots $\rightsquigarrow$ ICERM semester visitor Christian Himpe!
- MOR methods for discrete-time systems.
- Extensions to descriptor systems $E\dot{x} = Ax + Bu$, E singular.
- Parametric model reduction:

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x,$$

  where $p \in \mathbb{R}^d$ is a free parameter vector; parameters should be preserved in the reduced-order model.

G. Obinata and B.D.O. Anderson.
Model Reduction for Control System Design.
Springer-Verlag, London, UK, 2001.

P. Benner, E.S. Quintana-Ortí, and G. Quintana-Ortí.
State-space truncation methods for parallel model reduction of large-scale systems.
Parallel Comput., 29:1701–1722, 2003.

P. Benner, V. Mehrmann, and D. Sorensen (editors).
Dimension Reduction of Large-Scale Systems.
Lecture Notes in Computational Science and Engineering, Vol. 45, Springer-Verlag, Berlin/Heidelberg, 2005.

A.C. Antoulas.
Approximation of Large-Scale Dynamical Systems.
SIAM Publications, Philadelphia, PA, 2005.

P. Benner.
Numerical linear algebra for model reduction in control and simulation.
GAMM Mitteilungen 29(2):275–296, 2006.

W.H.A. Schilders, H.A. van der Vorst, and J. Rommes (editors).
Model Order Reduction: Theory, Research Aspects and Applications.
Mathematics in Industry, Vol. 13, Springer-Verlag, Berlin/Heidelberg, 2008.

P. Benner, J. ter Maten, and M. Hinze (editors).
Model Reduction for Circuit Simulation.
Lecture Notes in Electrical Engineering, Vol. 74, Springer-Verlag, Dordrecht, 2011.

U. Baur, P. Benner, and L. Feng.
Model order reduction for linear and nonlinear systems: a system-theoretic perspective.
Archives of Computational Methods in Engineering 21(4):331–358, 2014.

P. Benner, A. Cohen, M. Ohlberger, and K. Willcox (editors).
Model Reduction and Approximation: Theory and Algorithms.
SIAM Publications, Philadelphia, PA, 2017.